Space, Uncertainty and Intergenerational Issues
Essays in Environmental Economics

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Contents

Introduction

This brief introduction presents the research questions that will be addressed in the four chapters of this thesis. More detailed introductions will be available at the beginning of each chapter. The red line connecting the chapters is the attempt to introduce some aspects of complexity in the traditional literature of environmental and resource economics. A manifesto on the vision of social-ecological systems as complex adaptive systems can be found in Levin et al (2012): nonlinear feedbacks, non convexities, strategic interactions, individual and spatial heterogeneity, varying time scales and stochasticity are key factors that cannot be ignored in the analysis of the complex coupling between the economic and environmental side of our ecosystem. Some of the these key factors will be considered in the models presented in the remainder of this work. The first chapter proposes a very simple model of the joint dynamics of capital and pollution, where capital stands for an homogeneous good that can be produced and consumed, while pollution accounts for the environmental degradation. Pollution is a by-product of production and in the meantime negatively affects production via a multiplicative damage function, as in framework of the Integrated Assessment Models proposed by Nordhaus (1992) and Nordhaus (2008). In these models a functional form for the damage function is assumed and the parameters are estimated (a literature review about the different functional forms proposed for the damage function is presented in Ortiz and Markandya, 2010). The stronger the effect of damage function, the less satisfactory the outcome of the economic activity. In principle the damage function can drive the economic outcome to zero only if its denominator assumes an infinite value. This is due to the choice of a Cobb-Douglas production function. In this chapter is shown that assuming an S-shaped function a la' Skiba (1978), multiple equilibria arise and depending on the initial conditions of capital and pollution an economy can be condemned to a poverty trap.

The second chapter, Pollution Diffusion and Abatement Activities across Space and over Time, builds on the first , introducing space into the picture. This paper belongs to a relatively young stream of literature that bridges the gap between new economic geography (see, for example, Fujita 1999) and growth theory (see Boucekkine, 2009). The key assumption is the continuity of space and the possibility of capital and pollution to flow across space thanks to a diffusion-like mechanism. In particular this chapter focuses on the role that capital and pollution diffusion has in shaping the basins of the attraction of the equilibria when a convex-concave production function enters the stage. Diffusion of capital can be either beneficial or detrimental, depending on its intensity and on the initial allocation of capital across space. The results are based on numerical simulation and the complexity of the problem requested an ad hoc numerical algorithm for the solutions to be displayed.

The third chapter, Sustainability and Intertemporal equity: a Multicriteria approach, deals with sustainability and intertemporal equity issues through the lenses of a multicriteria approach. In most situations a decision maker have different conflicting criteria to meet: this heterogeneity of the goals is particularly striking when sustainability and long run economic growth have to be taken into account simultaneously. The Discounted Utilitarianism and Green Golden Rule are two social welfare functions, id est two conflicting welfare criteria, that rely on two different normative approaches to the most debated question about what should be considered the appropriate form for a long run social welfare function (see Heal). The chapter tries to provide an answer to this question, by evaluating which welfare criteria yields the best outcome; the Chichilnisky criterion (see Chichilnisky,) proves to be a useful unifying framework to interpret the problem.

In the fourth and last chapter, Pollution Control under Uncertainty and Sustainability Concern, there is an analysis about the implications of environmental policy on pollution in a stochastic framework with finite horizon and sustainability concern. The idea of minimizing the negative externalities of pollution, taking into account both the damages pollution brings in the short run and the heritage pollution leaves at the end of the time span, echoes the Chichilnisky criterion cited before. The approach has been made more adherent to reality thanks to the stochastic differential equation that describes the evolution of pollution along time.

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Chapter 1

Damage Functions and Green Poverty Trap

1.1 Introduction

As the U.S. federal agency NOAA, National Oceanic and Atmospheric Administration has recently pointed out, 10 out of the last 15 years have been the hottest since reliable temperature misurations started, at the end of the 19th century. Nowadays it has been widely accepted that this global rising of temperature has an anthopogenic cause. It is the same agency who provides quarterly data about the Carbon Dioxide concentration at a global level. Putting together direct and indirect misurations, these data inequivocably show the exponential growth Carbon Dioxide concentration has been through in the last 140 years. The bright side of this story is the unprecedented global economic growth the world has experienced in the same span of time. GDP - Gross Domestic Product - calculations go back some decades, but through indirect misurations, economists have been able to build reliable estimates of the path this proxy of economic growth has followed: similarly to the Carbon Dioxide concentration, and roughly simultaneously, the GDP has performed an exponential growth. Are these phenomena related? Are growth and CO_2 concentration doomed to proceed hand by hand? Is technology eventually going to inverse this tendency, making available greener and greener production alternatives? Or, on the contrary, do scarcity constraints on exhaustible resources, overexploitation of renewable resources and irreversibility triggered by pollution pose an ominous threat to the long run growth? This is a fiercely open debate in the environmental-resource economic literature. Early concerns about the joint evolution of economic growth and pollution can be traced back to the 70s and early 80s, with the work of Maler (1974), Brock (1973) and Luptacik (1982), among many others [1]. Relying on the idea that growth and pollution are two sides of the same coin, integrated models have been proposed, where environmental pollution has been introduced both as an imput and as a by-product of production, in the framework of neoclassical growth models. In particular, the idea behind pollution as an input of production is that the more pollution is allowed, the less costly are the techniques of

[1] In this regard, a complete and precious literature review can be found in Xepapadeas (2005)

production (see, for examples, Bovemberg (1995), Smulders (1999), Mohtadi (1996), Rubio (2000)). Alternatively, the effect of pollution as a by-product of production has been studied through a damage function, as in Nordhaus (1992) and Nordhaus (2008) or, more recently in Brechet (2014), Bondarev (2013), Anita (2013).

This paper is an attempt to build on the latter stream of literature, by considering pollution as a negative production externality, whose effects can be modeled by a damage function. Let aside Anita (2013), the papers in the area of integrated (economy-environment) assessment models employ an aggregate Cobb-Douglas production function. The consequences of this choice are twofold: there exists only one asymptotically stable equilibrium and the (multiplicative) damage function can drive the economy to the collapse only if its denominator takes infinite value. This amounts to say that however hard the damage may be, there are no (green) poverty trap, no matter the initial conditions of the integrated system economy-environment. This is barely an acceptable semplification of reality in two respects: history, or initial conditions, usually matter, and the damage function has to somehow take into account the non-infinite resilience of the environment. The most related paper is Anita (2013): with respect to this work, we do not consider pollution diffusion. We instead focus on the stability analysis of the equilibria both in a neoclassical and in a convex-concave production technology context. The main contribution is about the analysis of the consequences the saving ratio, the abatement activities, the damage function and environmental inefficiency exert on the long run performance of the economy and on the accumulation of pollution. In a nutshell, the model we propose comprises a module describing the evolution of capital and a module depicting the accumulation of pollution. Pollution is a by-product of production and, simultaneously, a detrimental factor for productivity.

The paper procedes as follows: section 2.2 presents the model in its general form, while sections 1.3, 1.4 and 1.5 decline the three production technologies AK, Cobb-Douglas and S-shaped respectively, delving into the existence and the stability analysis. Section 1.6 concludes.

1.2 The Model

The dynamic model is summarized by the following system of ordinary differential equations:

$$\begin{aligned} \frac{dk(t)}{dt} &= \frac{sf[k(t)][1-\tau]}{d[p(t)]} - \delta_k k(t) && (1.1) \\ &= \frac{sf[k(t)][1-u]^{\epsilon}}{1+bp(t)} - \delta_k k(t) \\ \frac{dp(t)}{dt} &= \theta[1-u]f[k(t)] - \delta_p p(t). && (1.2) \end{aligned}$$

Equation (2.1) represents the evolution of capital. The engine of capital accumulation lies in the production technology $f[k(t)]$: pollution does not enter directly the production function, but it affects the amount of output through the damage function $d[p(t)]$. Only a portion s of the produced output goes to capital investments. Moreover, a share τ of these investments is devoted to an environmental tax. Hence the interaction and the balancing between the production function $f[k(t)]$, the damage function $d[p(t)]$, the saving ratio s and the taxation rate τ, shapes the path of capital over time. Three different specifications of the production technology $f[k(t)]$ will be considered. First we will analyze an AK production technology, $f[k(t)] = \alpha k$. Then we will move to the neoclassical Cobb-Douglas production function, $f[k(t)] = \alpha k^\gamma$, and finally we will treat the case of a convex-concave production function, as in Skiba (1978). The damage function has been chosen as $d[p(t)] = 1 + b[p(t)]$, with $b > 0$: this particular formulation states that the pollution externality on production is null when pollution is absent, while production falls proportionally when pollution increases. Environmental taxation has been taken into account via the parameter τ: the tax revenue is $r(t) = \tau s f[k(t)]$, a portion of the capital investments. We assume that the goverment wants to mantain a balanced budget at any point in time, so the amount of resources collected through the taxation is totally devoted to abatement activities, $a(t)$. These abatement activities cut a certain share of pollution, $u \in (0, 1)$, by addressing a certain amount of not consumed output to this end. The associated cost is $a(t) = \mathcal{C}[u]sf[k(t)]$, where $\mathcal{C}(\cdot)$ is the cost function. By equating the tax revenue and abatement we conclude that $\tau = \mathcal{C}[u] = 1 - [1 - u]^\epsilon$ with $\epsilon \geq 1$, where the cost function is assumed to take the suitable form proposed by Bartz and Kelly (2008). The linear depreciation $\delta_k k$ follows the standard approach of the Solow model.

Equation (2.2) models the evolution of pollution. Production generates emissions which increase linearly the stock of pollution and $\theta > 0$ measures the degree of environmental inefficiency of economic activities. The abatement activities reduce a share u of emissions, thus $1 - u$ represents unabated emissions. For the sake of simplicity, in a model that already includes non-linearities and non-convexities, the depreciation of pollution, that is the self-cleaning capacity of the environment, is assumed to be proportional to the stock of pollution, $\delta_p p$: this is a gross simplification, given that the coefficient δ turns out to be stock dependant actually, that is $\delta_p = \delta_p(p)$, suffice it to mention the oceans absortion rate of carbon dioxide, strongly dependent on the stock of carbon dioxide itself.

The red line connecting all the ideas of this paper goes through 4 key parameters: the saving ratio s, the abatement share u, the damage function parameter b and the degree of environmental inefficiency θ. Parameters s and u represent two frozen controls, meaning that they could be the result of a policy choice: hence it is interesting to evaluate how their variation affects the performance of the economy. As for b and θ, they convey information about how hard pollution beat on production, in two different ways: via b, the damage function acts directly, simultaneously on the produced output, worsening the efficiency of the production technology - decrease in output produced from the same amount of imputs[2] - while θ influences production indirectly through the quantity of non-abated emissions that fosters that stock of pollution. But the real difference between b and θ is conceptual. Indeed, b gathers the overall response of the environment to the pressures of the economic growth and, as a consequence, b cannot be controlled by a policy or a technology, unless these countermeasures act against the accumulation of pollution itself: the exogeneity of b is constitutive. In regard to this, it is worth to note that our assumption about the deterministic, linear and continous dependance between the damage function and the stock of pollution is quite strong ($d[p(t)] = 1 + b[p(t)]$): it is admissible to imagine b dependent on p in some irregular and even stochastic way (for example via a jump process). Was this the case, a certain level of pollution stock could turn on some uncontrolled irreverisility into the environmental economics model 2.1 - 2.2. On the other hand, the parameter θ sums up the cleaning technology status: a technology effort in this direction could make θ decrease. The exogeneity of θ is not constitutive.

[2] It is possible to figure this out another way: imagine that the production technology is not affected, but the level of imput gets smaller: before even entering the production process a portion of the inputs becomes anavailable. The net result is the same

1.3 The AK scenario

In the case of the AK production function the system 2.1 - 2.2 becomes

$$\frac{dk(t)}{dt} = \frac{s\,[1-u]^{\epsilon}\alpha k(t)}{1+b\,p(t)} - \delta_k k(t). \tag{1.3}$$

$$\frac{dp(t)}{dt} = \theta\,[1-u]\,\alpha k(t) - \delta_p p(t). \tag{1.4}$$

Even in this relatively simple scenario, it is not possible to come up with a closed form solution for the complete dynamics. Nevertheless, a full description of the equilibria and their stability will be provided.

Proposition 1. *If the following condition on the parameters holds*

$$\alpha\, s\,[1-u]^{\epsilon} - \delta_k > 0 \tag{1.5}$$

then, the system 1.3 - 1.4 has two equilibria:

$$k_{eq} = 0. \tag{1.6}$$

$$p_{eq} = 0. \tag{1.7}$$

$$k^{eq} = \frac{(\alpha\, s\,[1-u]^{\epsilon} - \delta_k)\delta_p}{\theta b\alpha[1-u]\delta_k}. \tag{1.8}$$

$$p^{eq} = \frac{\alpha\, s\,[1-u]^{\epsilon} - \delta_k}{b\delta_k}. \tag{1.9}$$

In particular $0 = k_{eq} < k^{eq}$. *The equilibria* (k_{eq}, p_{eq}) *and* (k^{eq}, p^{eq})*, specified in 1.6 - 1.7 and 1.8 - 1.9, are unstable and stable respectively.*

Proof. See Appendix 1.7. ■ The condition in 1.5 coincides with the condition on (k^{eq}, p^{eq}) to be in the strictly positive orthant $\mathbb{R}^2_{++}$: if a non trivial solution exists then its local stability is guaranteed. On the one hand it is clearly reasonable that in an AK framework the net investments in capital accumulation have to be stronger than the capital depreciation for the economy to survive. On the other hand instead, taking into account the dynamics of pollution and its effects on production via the damage function adds transitional dynamics to the original model, at the expense of endogeneous growth: it is well known that the AK model is the simplest example of endogeneous growth model but lacks of any dynamics. As for the trivial solution $(k_{eq}, p_{eq}) = (0, 0)$, it is easy to show that it has a sort of stable manifold superimposed on the k axis $(0, p)$, but it is not attractive anywhere else.

1.3.1 Comparative statics

The analytical expression of the steady state allows for some comparative statics. As underlined in section 2.2, we choose to focus on s, u, b and θ. Here it follows the analysis:

$$\frac{\partial k_{eq}}{\partial s} = \frac{[1-u]^{\epsilon-1}\delta_p}{\delta_k\, b\,\theta}, \tag{1.10}$$

$$\frac{\partial p_{eq}}{\partial s} = \frac{\alpha\,[1-u]^{\epsilon}}{\delta_k\, b}, \tag{1.11}$$

$$\frac{\partial k_{eq}}{\partial u} = -\frac{(\alpha\, s\,([1-u])^{\epsilon}\,(\epsilon-1)+\delta_k)\,\delta_p)}{\delta_k\, b\,\theta\,\left([1-u]\right)^2\alpha}, \tag{1.12}$$

$$\frac{\partial p_{eq}}{\partial u} = -\frac{\alpha\, s\,([1-u])^{\epsilon-1}\,\epsilon}{\delta_k\, b}, \tag{1.13}$$

$$\frac{\partial k_{eq}}{\partial b} = -\frac{(\alpha\, s[1-u]^{\epsilon}-\delta_k)\,\delta_p}{\delta_k\, b^2\,\theta\,[1-u]\,\alpha}, \tag{1.14}$$

$$\frac{\partial p_{eq}}{\partial b} = -\frac{\alpha\, s[1-u]^{\epsilon}-\delta_k}{\delta_k\, b^2}, \tag{1.15}$$

$$\frac{\partial k_{eq}}{\partial \theta} = -\frac{(\alpha\, s[1-u]^{\epsilon}-\delta_k)\,\delta_p}{\delta_k\, b\,\theta^2[1-u]\,\alpha}, \tag{1.16}$$

$$\frac{\partial p_{eq}}{\partial \theta} = 0. \tag{1.17}$$

s) An increasing in the saving ratio improves the results of the economy, given that the partial derivate in 1.10 is always positive. Unfortunately this occurrence has the side effect of increasing the steady state stock of pollution, as it is clear from 1.11: more in detail, the ratio between the derivative $\frac{\partial k_{eq}}{\partial s}/\frac{\partial p_{eq}}{\partial s} = \frac{\delta_p}{\theta\,[1-u]\,\alpha}$ shows that the greater is the share of abated emission u, the wider grows the spread between k_{eq} and p_{eq}: k^{eq} and k^{eq} both grow, but the more u increases, the wider is the difference between their rate of growth.

u) As it is evident from equation 1.12, under the hypotheses on the parameters, the overall derivative is always negative. An increase in the abatement share u drains away resources from the capital accumulation investments, worsening the performance of the economy. Similarly, the equilibrium level of the pollution stock decreases as the share of abated emission goes up, says equation 1.13.

b) Observing equation 1.14, it is clear that the tougher the damage function is, the worse the final outcome of the economy will be. In our model, be it the AK or the Cobb-Douglas or the S-shaped version, k^{eq} and p^{eq} are proportional via a contant that does not comprise b, so it is no wonder that a marginal change in b affects k^{eq} and p^{eq} the same way.

θ) The behaviour of the steady state of the economy k^{eq} in response to an increase in the degree of environmental inefficiency θ is similar to the one following an increase of b: θ and k^{eq} are negatively related, as shown in 1.16. Conversely, the final pollution stock is not affected by variations of θ, given that k_{eq} is proportional to $1/\theta$ and p_{eq} is proportional to k_{eq} via a factor θ.

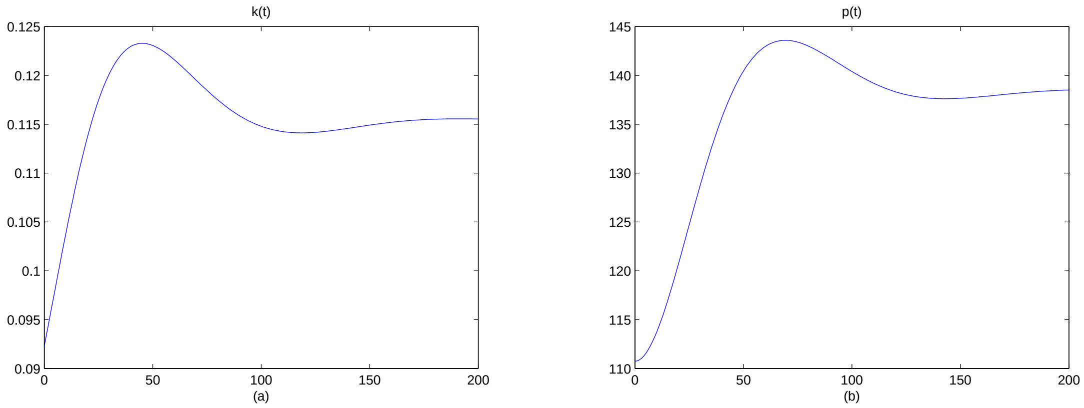

Figure 1.1: Dynamics of k (left) and p (right), AK technology

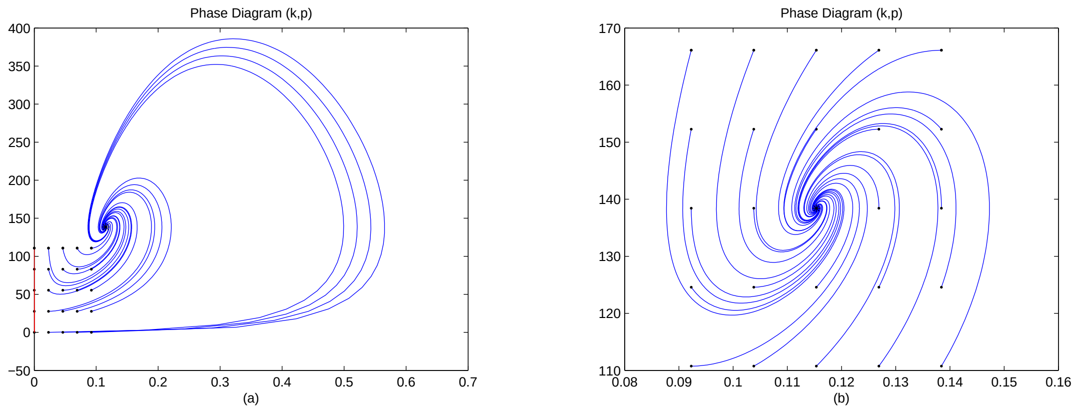

Figure 1.2: Global phase diagram of (k,p) (left), and particular of the stable equilibrium, (right).

1.3.2 Numerical example

In our numerical example the following list of parameters value will be used:

$$\begin{cases} s = 0.15, \;\; u = 0.4, \;\; \delta_k = 0.05, \;\; \delta_p = 0.05 \\ \theta = 1, \;\; b = 1, \;\; \alpha = 100, \;\; \epsilon = 1.5 \end{cases} \tag{1.18}$$

The value of the saving ratio $s = 0.15$ is the mean of the saving ratio percentages of gdp provided by data on OECD countries (see ...), while for $u = 0.4$ we employed the targets for reducing the emission of GHGs discussed in the Global Greenhouse Gas Abatement Cost Curve prepared by McKinsey and Company, (see the Reference Section). The depreciation rate of capital $\delta_k = 0.05$ is in line with standard values for Solow-type models, while the pollution decay rate, δ_p, is set equal to 0.05 (see, for example Saltari e Travaglini, 2014). The parameter b and θ are set to 1 for the sake of simplicity, as well as for $\epsilon = 1.5$, whose only condition is to be greater than or equal to 1. Finally the scaling factor α is assumed to be 100, here and in the remainder of the paper, as a matter of consistency.

The results of the simulations are shown in Figures 1.1, 1.2 and 1.3. The dynamics of k and p are presented in the first Figure, where it is possible to note that both k and p reach a steady state, after some evolution along time. Panel (a)

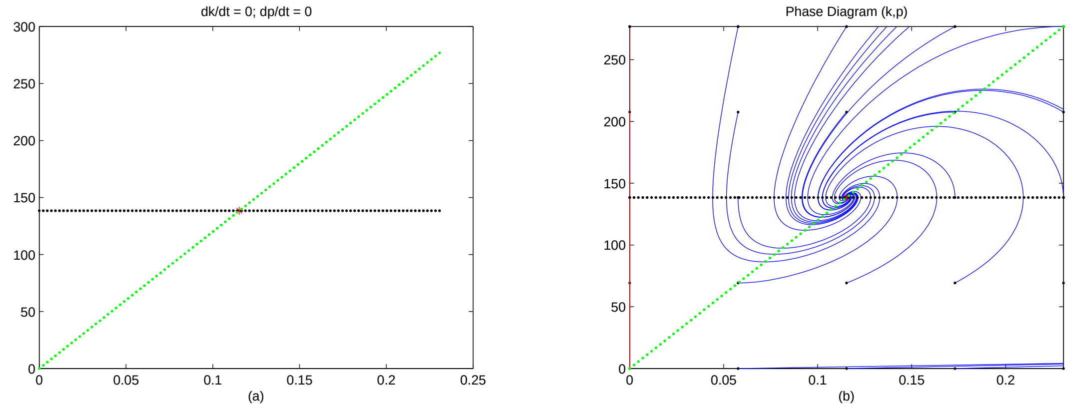

Figure 1.3: On the left the curves $dk/dt = 0$ and $dp/dt = 0$ in black and green respectively; on the right the overall picture

and panel (b) of Figure 1.2 depict the phase diagrams of the system 1.3 - 1.4 around the two equilibria (k_{eq}, p_{eq}) and (k^{eq}, p^{eq}) respectively: these are not the usual phase diagrams with arrows showing the directions of the trajectories in every region of interest, because we preferred to show the entire path of each trajectory stemming from points regularly spread around each equilibrium. We think that this way of visualizing the phase diagram has more explanatory power when the eigenvalues associated to the equilibria are complex and counjugate, so that the trajectories show spiral-shaped behaviour. Panel (a) describes the time evolution of trajectories starting from point around the origin. It is clear that only the trajectories whose outset lies on the k axis, i.e. $(k_{t=0}, p_{t=0}) = (0, p)$ (depicted in red), are doomed to die out into the origin: only if the economy is absent at the beginning, the final outcome is $(k_{eq}, p_{eq}) = (0, 0)$. As shown in the appendix 1.7, the k axis resembles the stable manifold of a saddle point equilibrium. Outside this axis, all the trajetories are attracted by the upper equilibrium (k_{eq}, p_{eq}), as it is observable both in panel (a) and (b) of the same Figure. In the latter panel, the detail of trajectories around (k_{eq}, p_{eq}) is shown: in the Appendix 1.7 it is demonstrated that if this equilibrium exists, then the eigenvalues associated to the Jacobiam matrix are complex and conjugate, giving rise to the spiral-shaped behaviour. Finally in Figure 1.3 the overall story is on the stage: on the left the curves $dk/dt = 0$ and $dp/dt = 0$ in black and green respectively, with their intersection point (k^{eq}, p^{eq}), the red asterix; while on the right the complete phase diagram with $dk/dt = 0$ and $dp/dt = 0$ superimposed.

1.4 The Cobb-Douglas scenario

In the case of the Cobb-Douglas production function the system 2.1 - 2.2 reads as:

$$\frac{dk(t)}{dt} = \frac{s\,[1-u]^{\epsilon}\,\alpha k(t)^{\gamma}}{1 + b\,p(t)} - \delta_k k(t). \tag{1.19}$$

$$\frac{dp(t)}{dt} = \theta\,[1-u]\,\alpha k(t)^{\gamma} - \delta_p p(t). \tag{1.20}$$

In line with the neoclassical concave Cobb-Douglas production function it must be that $0 < \gamma < 1$: the inclusion of this new parameter complicates the analysis of the steady states.

Proposition 2. *The system 1.19 - 1.20 has two equilibria:*

$$k_{eq} = 0. \tag{1.21}$$

$$p_{eq} = 0. \tag{1.22}$$

$$k^{eq} = RootOf\left(k^{\gamma}\alpha\, b\, \delta_k\, \theta\,(1-u) - \alpha\, k^{\gamma-1} s\,(1-u)^{\epsilon}\delta_p + \delta_k\, \delta_p\right). \tag{1.23}$$

$$p^{eq} = \frac{\theta\,(1-u)\,\alpha\, {k_{eq}}^{\gamma}}{\delta_p}. \tag{1.24}$$

In particular $0 = k_{eq} < k^{eq}$*. The equilibria* (k_{eq}, p_{eq}) *and* (k^{eq}, p^{eq})*, specified in 1.21 - 1.22 and 1.23 - 1.24, are unstable and stable respectively.*

Proof. See Appendix 1.8. ■ It is comforting to note that k^{eq} specified in the equation 1.23 boils down to the solution of the AK case, i.e. equation 1.8, when $\gamma \to 1$. Even though it is not possible to provide a closed form expression for k^{eq} in 1.23 under the general framework $0 < \gamma < 1$, it is still manageable to get analytical solutions for the case $\gamma = 1/2$, without loosing generality [3]. For simplicity we assume $\delta_k = \delta_p = \delta$, and we obtain the following results:

$$k^{eq} = \frac{b\,\theta\,(1-u)^{1+\epsilon}\alpha^2 s - 1/2\,\delta\,\sqrt{\delta^2 + 4\,b\,\theta\,(1-u)^{1+\epsilon}\alpha^2\, s} + 1/2\,\delta^2}{b^2\theta^2(1-u)^2\alpha^2}. \tag{1.25}$$

$$p^{eq} = \frac{1}{2}\,\frac{\theta\,(1-u)\,\alpha\,\sqrt{2}}{\delta}\sqrt{\frac{2\,b\,\theta\,(1-u)^{1+\epsilon}\alpha^2 s - \delta\,\sqrt{\delta^2 + 4\,b\,\theta\,(1-u)^{1+\epsilon}\alpha^2 s} + \delta^2}{b^2\theta^2u^2\alpha^2}}. \tag{1.26}$$

In this particular setting, the number of parameters and the intricacy of the solutions still doesn't make prohibitive an attempt to perform the comparative statics analysis.

1.4.1 Comparative statics, $\gamma = 1/2$

In this section we propose a comparative statics analysis for the case $\gamma = 1/2$ and $\delta_k = \delta_p = \delta$. This additional restriction on the parameters will therefore allow us to unequivocally decide on the signs of the partial derivates. Given that k^{eq} and p^{eq} are proportional, via the positive constant $C = \theta\,(1-u)\,\alpha/\delta_p$, we omit to report the partial derivates of p^{eq} with respect to b and s: indeed the effect of these parameters on p^{eq} is only indirect, through k^{eq}. Despite the complexity of the following expressions, it is not difficult to understand whether the partial derivative is positive or negative. Let us define

$$\Delta_{CD} \equiv \delta^2 + 4\,b\,\theta\;(1-u)^{1+\epsilon}\,\alpha^2 s \tag{1.27}$$

[3] We could have chosen $\gamma = \frac{1}{3}$ both in the comparative statics and in the next numerical example section, but the expression of solution would have been too complicated, without adding any insight in the nature of the results

The comparative statics analysis is:

$$\frac{\partial k^{eq}}{\partial s} = \frac{1}{b^2\theta^2\,(1-u)^2\,\alpha^2}\left(\frac{b\,\theta\,(1-u)^{1+\epsilon}\alpha^2\left(\sqrt{\Delta_{CD}}-\delta\right)}{\sqrt{\Delta_{CD}}}\right), \tag{1.28}$$

$$\frac{\partial k^{eq}}{\partial u} = -\frac{(1-u)^{1+\epsilon}\,s\,(\epsilon-1)\left(\sqrt{\Delta_{CD}}-\delta\right)}{b\theta\,\sqrt{\Delta_{CD}}\,(1-u)^3} - 2\,\frac{\delta\,k^{eq}}{(1-u)\,\sqrt{\Delta_{CD}}}, \tag{1.29}$$

$$\frac{\partial p^{eq}}{\partial u} = -\frac{1}{2}\,\frac{\sqrt{2}\alpha\,s\,(1-u)^{-1+\epsilon}\,(\epsilon+1)\left(\sqrt{\Delta_{CD}}-\delta\right)}{\delta\,b\sqrt{\Delta_{CD}}}\,\frac{1}{\sqrt{\frac{2\,b\theta\,(1-u)^{1+\epsilon}\alpha^2 s-\delta\,\sqrt{\Delta_{CD}}+\delta^2}{b^2\theta^2(1-u)^2\alpha^2}}}, \tag{1.30}$$

s) The expression on the hand-right side of equation 1.28 is the product of two factors. The first one is surely positive. The term $\sqrt{\delta^2 + 4\,b\,\theta\,(1-u)^{1+\epsilon}\alpha^2 s} - \delta$, is clearly greater than 0: indeed the squared root is a monotonic function, so $\sqrt{x^2+\phi} > \sqrt{x} = x$ if ϕ is positive, as it is the case now, being $\phi = 4\,b\,\theta\,(1-u)^{1+\epsilon}\alpha^2 s$. Here it is the same result of the AK scenario: an increase in the capital investment is beneficial for the economy. It is interesting to note that the analogous result for the AK case has no dependance on s itself: with a Cobb-Douglas technology, the concavity of the production function complicates things in such a way that the marginal variation in the performance of the economy, in response to an increase in the saving ratio, depends on the saving ratio itself; in particular, the beneficial effect decreases as $s \to 1$, where it tends to its infimum.

u) It easy to note that derivative in 1.29 is always negative. As in the analogous context of the AK model, an increase in the share of abated emissions brings a worsening in the economic performance. As for the equilibrium level of pollution stock p^{eq}, on the right-hand side of equation 1.30 the are two factors, both positive. Indeed, the argument under the squared root at the denominator of the second factor is $2k^{eq}$, ergo positive. Hence, the overall derivative $\partial p^{eq}/\partial u$ is negative. The pollution stock diminishes when the environmental care activity becomes stronger: there is a trade-off between growth and environmental quality.

$$\begin{aligned}\frac{\partial k^{eq}}{\partial b} &= \frac{1}{b^2\theta^2\,(1-u)^2\,\alpha^2}\left(\frac{\theta\,s\,(1-u)^{1+\epsilon}\alpha^2\left(\sqrt{\Delta_{CD}}-\delta\right)}{\sqrt{\Delta_{CD}}}\right)\\ &\quad -2\,\frac{b\theta\,(1-u)^{1+\epsilon}\,\alpha^2 s - 1/2\,\delta\,\sqrt{\Delta_{CD}} + 1/2\,\delta^2}{b^3\theta^2\,(1-u)^2\,\alpha^2},\end{aligned} \tag{1.31}$$

$$\begin{aligned}\frac{\partial k^{eq}}{\partial \theta} &= \frac{1}{b^2\theta^2\,(1-u)^2\,\alpha^2}\left(\frac{b\,s\,(1-u)^{1+\epsilon}\alpha^2\left(\sqrt{\Delta_{CD}}-\delta\right)}{\sqrt{\Delta_{CD}}}\right)\\ &\quad -2\,\frac{b\theta\,(1-u)^{1+\epsilon}\,\alpha^2 s - 1/2\,\delta\,\sqrt{\delta^2+4\,b\theta\,u^{1+\epsilon}\alpha^2\Delta_{CD}\,s} + 1/2\,\delta^2}{b^2\theta^3\,(1-u)^2\,\alpha^2},\end{aligned} \tag{1.32}$$

$$\frac{\partial p^{eq}}{\partial \theta} = \frac{1}{2}\,\frac{\sqrt{2}\alpha\,s\,(1-u)^{\epsilon}\left(\sqrt{\Delta_{CD}}-\delta\right)}{b\theta\,\delta\,\sqrt{\Delta_{CD}}}\,\frac{1}{\sqrt{\frac{2\,b\theta\,(1-u)^{1+\epsilon}\alpha^2 s-\delta\,\sqrt{\Delta_{CD}}+\delta^2}{b^2\theta^2(1-u)^2\alpha^2}}}, \tag{1.33}$$

b) The damage function parameter b has the same effect both on k^{eq} and p^{eq}, as we explained before. After some manipulation of the expression in 1.31, whose particulars are in the Appendix 1.8, it turns out that $\partial k^{eq}/\partial b$ is negative. As in the AK scenario, the more damage pollution brings to capital investment, the less satisfactory is the long run behaviour of the economy, and, consequentely, the less pollution is eventually accumulated.

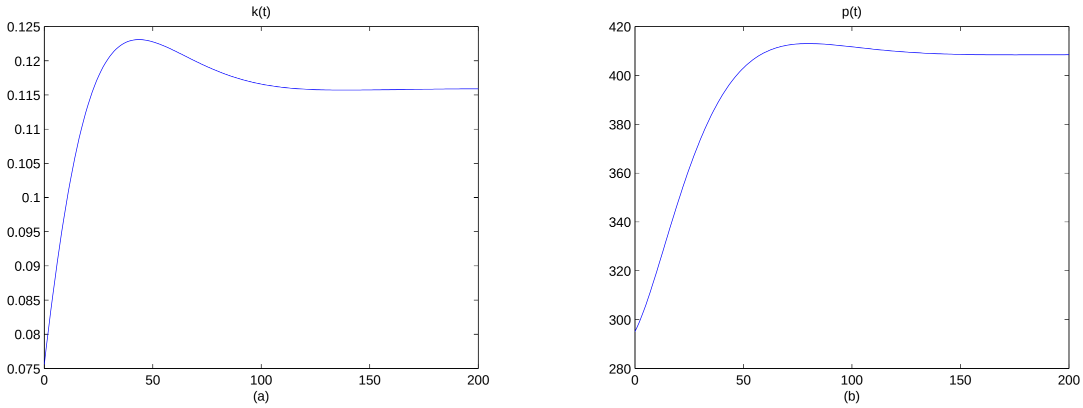

Figure 1.4: Dynamics of k (left) and p (right), Cobb-Douglas technology.

θ) The marginal effects of an increase in the environmental inefficiency θ on k^{eq} is negative, as it possible to elicit from the comparison between 1.31 and 1.32. Details in the Appendix 1.8. A look at equation 1.33 confirms that, differently from the Ak scenario, the increase in θ is negative for the environment, causing a greater long run value for the pollution stock: $\partial p^{eq}/\partial\theta > 0$. When a Cobb-Douglas production function models the agglomeration of economy, there is no more doubt: an improvement in environmental efficiency-related technologies, that is a decrease of θ, is beneficial for the long run of the economy and the enviroment.

1.4.2 Numerical example

The same set of parameters as section 1.3.2 will be employed, with the addiction of the capital share γ:

$$\begin{cases} s = 0.15,\ u = 0.4,\ \delta_k = 0.05,\ \delta_p = 0.05 \\ \theta = 1,\ b = 1,\ \alpha = 100,\ \epsilon = 1.5,\ \gamma = \frac{1}{2} \end{cases} \tag{1.34}$$

The choice of the capital share $\gamma = \frac{1}{2}$ was suggested by a consistency argument with respect to the comparative statics section. The simulations are displayed in Figures 1.4, 1.5 and 1.6. It can be noticed that the results are quite similar to the ones of the AK scenario, so we will proceed quickly. Figure 1.4 shows that the dynamics of k and p : eventually they reach their steady state (k^{eq}, p^{eq}), as in equations 1.25 - 1.26 . The phase diagrams of Figure 1.5 tell that the trivial equilibrium can be reached only with initial condition $(0, p)$, while (k^{eq}, p^{eq}) is a stable equilibrium with complex and conjugate eigenvalues. The intepretation of the two panels of Figure 1.6 is straighforward: panel a) shows the loci $dk/dt = 0$ and $dp/dt = 0$ in black and green respectively, while panel b) sums up all the information in one picture. The comparison between the a) panels of Figure 1.3 and Figure 1.6 shows in which sense the AK case is the limit of the Cobb-Douglas case when $\gamma \to 1$: the black line bends to become flat and the green one looses its concavity.

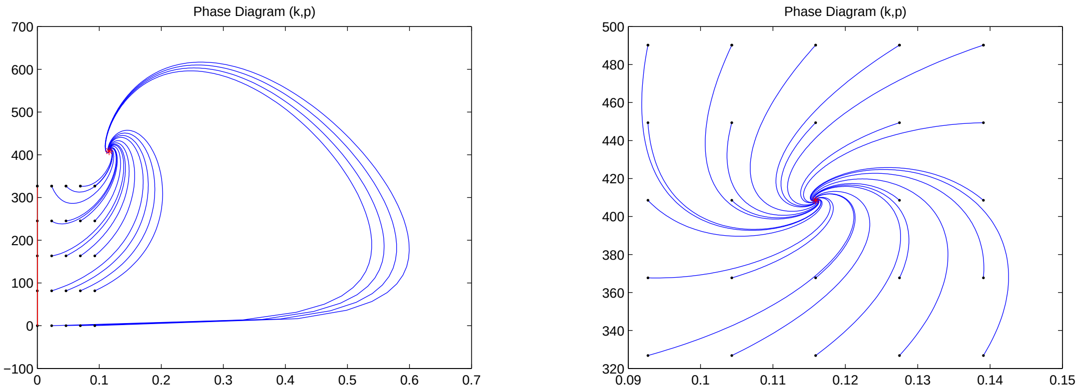

Figure 1.5: Global phase diagram of (k,p) (left), and particular around the stable equilibrium (right).

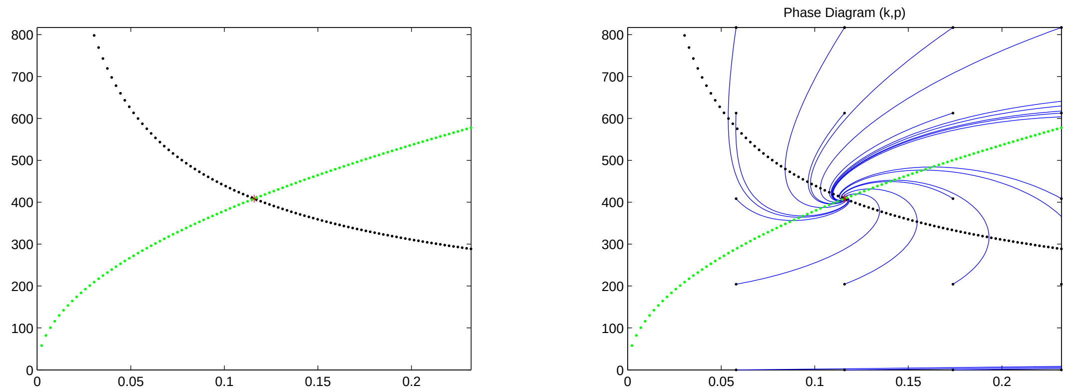

Figure 1.6: On the left the curves $dk/dt = 0$ and $dp/dt = 0$ in black and green respectively; on the right the overall picture

1.5 The S-shaped scenario

Now it is the turn of the S-shaped technology. Before we proceed in the analysis of the system 2.1 - 2.2, it is useful to present the S-shaped production function in its basic framework, in order to highligth its main characteristics: these are well known results from the work of Skiba (1978), but it is nevertheless helpful to present them for clarity sake. The S-shaped production function reads as:

$$f[k(t)] := \frac{\alpha_1 k(t)^q}{1 + \alpha_2 k(t)^q},\ q > 1, \tag{1.35}$$

The interesting properties of $f[k(t)]$ in 1.35 are guaranteed if $\alpha_1 > 0$, $\alpha_2 > 0$. For simplicity, from now on we consider $q = 2$ and $\alpha_1 = \alpha_2 = \alpha > 0$. The S-shaped production function is not a neoclassical concave production function. It is both convex and concave: it was introduced in the attempt to take into account the possibily of poverty traps in which underdeveloped countries may be trapped, notwithstanding their efforts to grow. The law of motion of capital

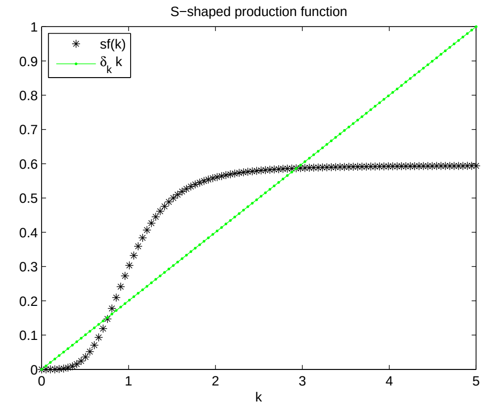

Figure 1.7: S-shaped production function

in the presence of this kind of technology is the following:

$$\frac{dk(t)}{dt} = \frac{s\alpha k(t)^2}{1+\alpha k(t)^2} - \delta_k k(t). \tag{1.36}$$

As a corollary of the next proposition we will show that, depending on the sign of $\Delta = \alpha^2 s^2 - 4\,\alpha\,\delta^2$, the previous differential equation has one or three equilibria. If $\Delta < 0$, the only equilibrium, globally asymptoticaly stable, is the origin, $k_{eq} = 0$, and the paper stops here. If $\Delta > 0$ there exist three equilibria: $0 = k_{eq} < k_{th} < k^{eq}$, stable, unstable and stable, respectively. The long run achievements of the economy depend on the value of the initial capital k_0: if this initial capital lies above the poverty trap threshold k_{th} the economy will eventually reach the upper steady state k^{eq}, otherwise, the economy will be trapped and its final outcome will be the collapse, $k \to k_{eq} = 0$. No dynamics instead if $k_0 \equiv k_{th}$: the threshold is the unstable equilibrium of the economy. Figure 1.7 gives an illustrative example. What we do next is studying how these results may be affected after the inclusion of the environmental perspective proposed in the system 2.1 - 2.2. In particular we would like to shed some light on the effect that pollution, via the damage function, may have on the existence and position of the poverty trap threshold. Equations 2.1 - 2.2 become :

$$\frac{dk(t)}{dt} = \frac{s[1-u]^{\epsilon}\alpha k(t)^2}{(1+bp(t))(1+\alpha k(t)^2)} - \delta_k k(t). \tag{1.37}$$

$$\frac{dp(t)}{dt} = \frac{\theta[1-u]\alpha k(t)^2}{1+\alpha k(t)^2} - \delta_p p(t). \tag{1.38}$$

Proposition 3. *Suppose $\epsilon = 1$. If the following condition on the parameters holds*

$$\alpha^2 {\delta_p}^2 s^2 (1-u)^2 - 4\,\alpha\, b\, {\delta_k}^2 \delta_p\, \theta\,(1-u) - 4\,\alpha\,{\delta_k}^2 {\delta_p}^2 > 0 \tag{1.39}$$

the system 1.37 - 1.38 has three equilibria:

$$k_{eq} = 0. \tag{1.40}$$

$$p_{eq} = 0. \tag{1.41}$$

$$k_{th} = \frac{1}{2}\frac{\alpha\,\delta_p\,s(1-u) - \sqrt{\alpha^2{\delta_p}^2 s^2(1-u)^2 - 4\,\alpha\,b\,{\delta_k}^2\delta_p\,\theta\,(1-u) - 4\,\alpha\,{\delta_k}^2{\delta_p}^2}}{\alpha\,\delta_k\,\left(b\,\theta\,(1-u)+\delta_p\right)}. \tag{1.42}$$

$$p_{th} = \frac{\theta\,(1-u)\,\alpha\,{k_{th}}^2}{\left(\alpha\,{k_{th}}^2+1\right)\delta_p}. \tag{1.43}$$

$$k^{eq} = \frac{1}{2}\frac{\alpha\,\delta_p\,s(1-u) + \sqrt{\alpha^2{\delta_p}^2 s^2(1-u)^2 - 4\,\alpha\,b\,{\delta_k}^2\delta_p\,\theta\,(1-u) - 4\,\alpha\,{\delta_k}^2{\delta_p}^2}}{\alpha\,\delta_k\,\left(b\,\theta\,(1-u)+\delta_p\right)}. \tag{1.44}$$

$$p^{eq} = \frac{\theta\,(1-u)\,\alpha\,{k^{eq}}^2}{\left(\alpha\,{k^{eq}}^2+1\right)\delta_p}. \tag{1.45}$$

In particular $0 = k_{eq} < k_{th} < k^{eq}$*. The equilibria* (k_{eq}, p_{eq}) *and* (k^{eq}, p^{eq}) *are stable, while* (k_{th}, p_{th}) *is unstable.*

Proof. See Appendix 1.9. ■ The presence of the environmental perspective does not rule out the possibility of the poverty trap. But clearly the damage function, via the parameter b, and the environmental inefficiency factor θ affect both the position of the poverty trap and the performance of the economy. More details in the next comparative statics section.

1.5.1 Comparative statics

In this section we present the usual comparative statics analysis with respect to s, u, b and θ, in the hypothesis of Proposition 3. To simplify, we assume that $\delta_k = \delta_p = \delta$: this choice makes calculations and exposition easier, without modifying the results. In what follows we define

$$\Delta_{Ss} \equiv \alpha\,\left(\alpha\,s^2\,(1-u)^2 - 4\,b\delta\,\theta\,(1-u) - 4\,\delta^2\right)$$

We display the analysis of the k variable only, not to flood the papers with too equations. We will focus on the opposite effects marginal changes in the parameters have on the equilibria (k_{th}, p_{th}) and (k^{eq}, p^{eq}).

$$\frac{\partial k_{th}}{\partial s} = \frac{1}{2}\frac{(1-u)\left(-\alpha\,s\,(1-u)+\sqrt{\Delta_{Ss}}\right)}{\sqrt{\Delta_{Ss}}\,(b\theta\,(1-u)+\delta)}, \tag{1.46}$$

$$\frac{\partial k^{eq}}{\partial s} = \frac{1}{2}\frac{(1-u)\left(\alpha\,s\,(1-u)+\sqrt{\Delta_{Ss}}\right)}{\sqrt{\Delta_{Ss}}\,(b\theta\,(1-u)+\delta)}, \tag{1.47}$$

$$\frac{\partial k_{th}}{\partial u} = -\frac{1}{2}\frac{\delta\,\left(-2\,b^2\theta^2 - \alpha\,s^2\,(1-u) - 2\,b\delta\,\theta + \sqrt{\Delta_{Ss}}s\right)}{\sqrt{\Delta_{Ss}}\,(b\theta\,(1-u)+\delta)}, \tag{1.48}$$

$$\frac{\partial k^{eq}}{\partial u} = -\frac{1}{2}\frac{\delta\,\left(+2\,b^2\theta^2 + \alpha\,s^2\,(1-u) + 2\,b\delta\,\theta + \sqrt{\Delta_{Ss}}s\right)}{\sqrt{\Delta_{Ss}}\,(b\theta\,(1-u)+\delta)}, \tag{1.49}$$

s) Under the hypothesis of Proposition 3, both k_{th} and k^{eq} are positive. So the term $\alpha\, s\,(1-u) \pm \sqrt{\Delta_{Ss}}$ is positive. This means that $\alpha\, s\,(1-u) > \sqrt{\Delta_{Ss}}$. Hence, looking at the partial derivatives in 1.46 and 1.47, it is easy to note that the effect of a marginal increase in the saving ratio is negative for k_{th} and positive for k^{eq}: devoting more effort in capital investment is beneficial both to improve the performance of the economy and to reduce the poverty trap threshold. The effects on the stock of pollution are similar: in the upper equilibrium (k^{eq}, p^{eq}) the long run level of pollution stock increase as well, underlining that a trade-off between growth and environmental care is still at stake in the S-shaped technology scenario. No matter the technology framework, a marginal increase in the saving ratio is always beneficial for the economy, but at the expense of environmental quality.

u) The effect of a marginal increase of the abated emission share u is described in equation 1.48 and 1.49. The key term is the paranthesis: $\sqrt{\Delta_{Ss}}s \pm 2\,b^2\theta^2 + \alpha\, s^2\,(1-u) + 2\,b\delta\,\theta$. Dividing it by s, it is immediate to note that the evaluation of the sign follows the same arguments as the previous case. Parameter u rising brings a negative effect on the performance of the economy and moreover increases the poverty trap threshold. Abating emissions comes at a cost: reduces the achievments of the economy and exposes to the risk of remaining trapped in a low capital cul de sac. Hence our model suggests that underdeveloped countries should think carefully about the investments in abatement activities.

$$\frac{\partial k_{th}}{\partial b} = -\frac{1}{2}\,\frac{\theta\,(1-u)\left(-\alpha\, s^2\,(1-u)^2 + 2\,b\delta\,\theta\,(1-u) + 2\,\delta^2 + \sqrt{\Delta_{Ss}}s\,(1-u)\right)}{\sqrt{\Delta_{Ss}}\,(b\theta\,(1-u)+\delta)^2}, \tag{1.50}$$

$$\frac{\partial k^{eq}}{\partial b} = -\frac{1}{2}\,\frac{\theta\,(1-u)\left(+\alpha\, s^2\,(1-u)^2 - 2\,b\delta\,\theta\,(1-u) - 2\,\delta^2 + \sqrt{\Delta_{Ss}}s\,(1-u)\right)}{\sqrt{\Delta_{Ss}}\,(b\theta\,(1-u)+\delta)^2}, \tag{1.51}$$

$$\frac{\partial k_{th}}{\partial \theta} = -\frac{1}{2}\,\frac{b\,(1-u)\left(-\alpha\, s^2\,(1-u)^2 + 2\,b\delta\,\theta\,(1-u) + 2\,\delta^2 + \sqrt{\Delta_{Ss}}s\,(1-u)\right)}{\sqrt{\Delta_{Ss}}\,(b\theta\,(1-u)+\delta)^2}, \tag{1.52}$$

$$\frac{\partial k^{eq}}{\partial \theta} = -\frac{1}{2}\,\frac{b\,(1-u)\left(\alpha\, s^2\,(1-u)^2 - 2\,b\delta\,\theta\,(1-u) - 2\,\delta^2 + \sqrt{\Delta_{Ss}}s\,(1-u)\right)}{\sqrt{\Delta_{Ss}}\,(b\theta\,(1-u)+\delta)^2}, \tag{1.53}$$

b) The overall effect of the damage function parameter b is detrimental: the performance of the economy decreases and the poverty trap threshold increases. Indeed, as a corollary of Proposition 3 it will be shown that

$$\alpha\, s^2\,(1-u)^2 - 2\,b\delta\,\theta\,(1-u) - 2\,\delta^2 > \sqrt{\Delta_{Ss}}s\,(1-u)$$

So, following a reasoning analogous to the one employed for s and u, the partial derivative in 1.50 is positive, while the one in 1.51 is negative. The parameter of the damage function b does not belong to an hypothetical set of control variables. As we mentioned earlier, this means that the only way for the policy maker to reduce its influence is to act indirectly through some pollution-reducing policy.

θ) The behaviour of the equilibria in response to marginal changes in the environmental inefficiency factor is the same as the previuos case of b, as a relatively close look at the relations in 1.52 and 1.53 can bring to light.

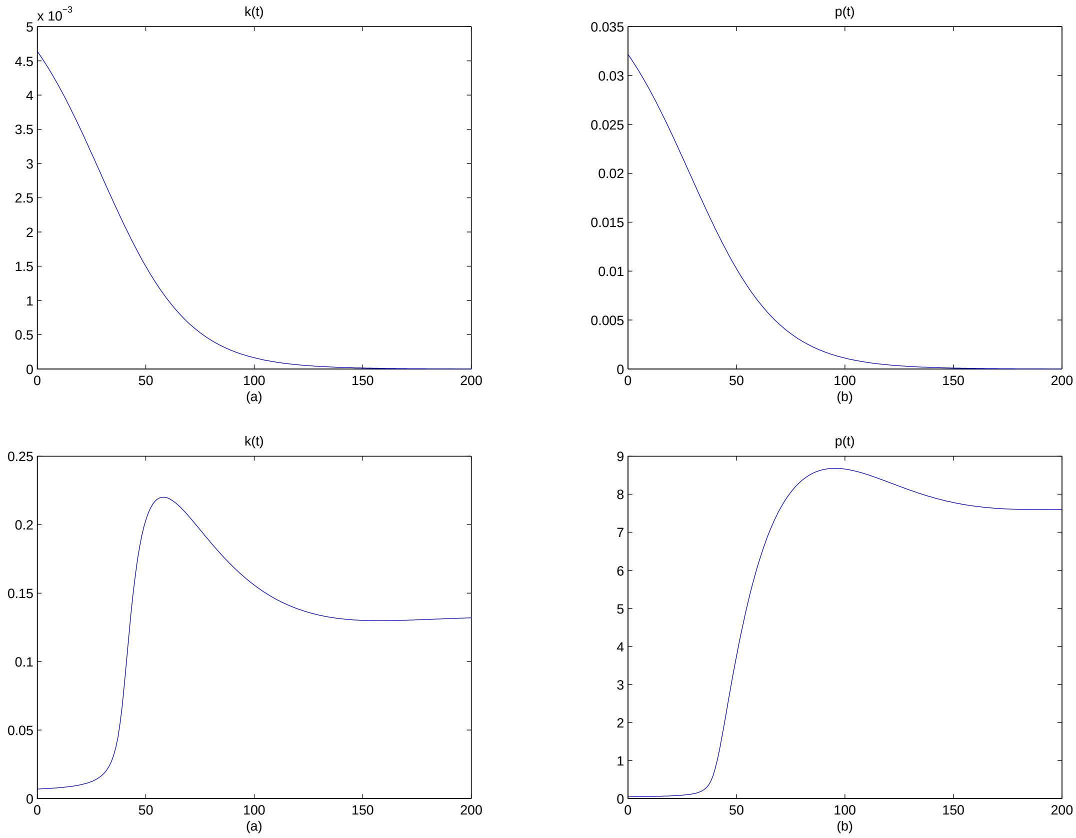

Figure 1.8: Dynamics of k (left) and p (right), S-shaped technology.

The interpretation of the results is different though. While the parameter b refers to the way a worsening of the environmental quality reduces the outcomes of production - usually not in control of the polluters, θ is imagined as largely dependent on technology related aspects. Hence, reducing this parameter can be beneficial in a twofold way: k^{eq} goes up, while k^{th} goes down.

1.5.2 Numerical example

The set of parameters we adopt for the simulations in the remainder of the paper is consistent with the hypothesis of Proposition 3:

$$\begin{cases} s = 0.15,\ u = 0.4,\ \delta_k = 0.05,\ \delta_p = 0.05 \\ \theta = 1,\ b = 1,\ \alpha = 100,\ \epsilon = 1. \end{cases} \tag{1.54}$$

The simulations are displayed in Figures 1.8, 1.9 and 1.10. The red line connecting the Figures is the role of the poverty trap threshold in shaping the dynamics and the steady state characteristics of k and p. The phase diagrams are built in the same way and with same reasons as the one displayed in the previous technology scenarios: no arrows, but the complete path of the some representative trajectories from the initial condition to the eventual long run outcome in the (k, p) plane. In addition, we will show some bifurcation Figures arising from the comparative statics discussion we did in the dedicated section. First we want to make a few comments about the complete dynamics of k and p illustrated in Figures 1.8. In the upper panels the dynamics of k, on the left, and p, on the right shows that the

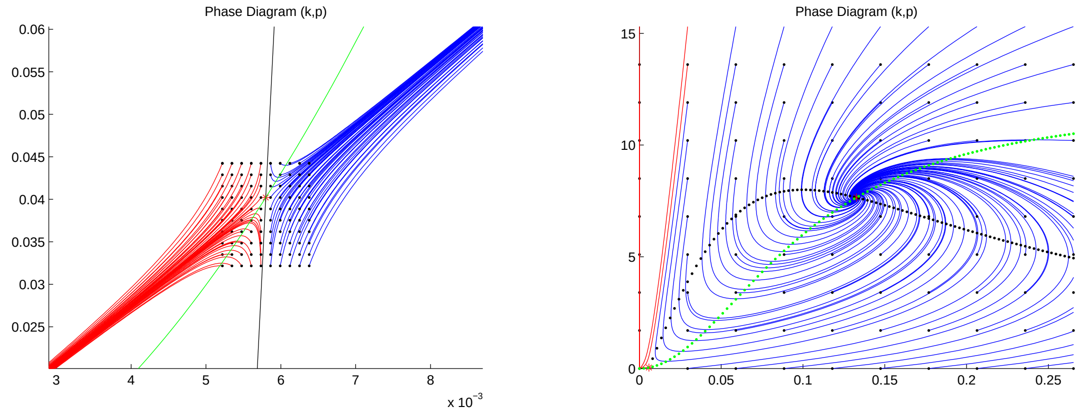

Figure 1.9: Particular of the phase diagram around the unstable equilibrium (left). Overall picture (right).

economy is condemned to fail when the initial conditions are such that the poverty trap threshold cannot be escaped. On the contrary, as shown in the bottom panels, a slight increase in the initial condition of capital could make the difference in guaranteeing a promising future to the economy. The dynamics alone cannot describe properly what is going on in (k, p) plane. Figure 1.9 tells an interesting story that could not have happened, was the technology framework AK or Cobb-Douglas. On the left, we can observe the paths of some trajectories departing from points equally spaced around the (k_{th}, p_{th}) unstable equilibrium. The point (k_{th}, p_{th}) lies at the first intersection between the loci $dk/dt = 0$ and $dp/dt = 0$, represented in green and black respectively. It is clear that the red trajectories belongs to the basin of the attraction of the origin: they will eventually reach the unpleasant equilibrium $(k_{eq}, p_{eq}) = (0, 0)$. On the other hand, the blue trajectories, stemming roughly on the right of the $dk/dt = 0$ locus, will move away from the origin. The right panel of the same Figure depicts the whole picture. It can be recognised the second intersection between the two loci $dk/dt = 0$ and $dp/dt = 0$, that is the upper equilibrium (k^{eq}, p^{eq}). All the blue trajectories belong to the basin of attraction of this equilibrium: they will eventually fall on it. It is interesting to note that the poverty trap threshold k_{th} depends on the initial pollution stock level: in other words, the same initial condition on capital may belong to the basin of attraction of the origin or to the basin of the upper equilibrium depending on the initial level of pollution stock. This characteristic can be detected in the Figure by looking at the red trajectories: they start more and more on the right, toward the increasing k direction, the more abundant is the initial level of pollution stock. Finally, via Figure 1.10, we provide a graphical representation of the main ideas expressed in the comparative statics section, through a bifurcation analysis. There are four panels. One for each of the four parameters considered before. Each panel has been built keeping every parameter frozen as in 1.54 but the one under scrutiny, which is free to vary in the intersection between its domain and the solution existence region, as in 1.39. In panel (a), $k^{eq} = k^{eq}(s)$ and $k_{th} = k_{th}(s)$ have been depicted, in green and black respectively. We see that there is minimum level of $s = s_{min}$ that guarantees the existence of two solutions: below this level the system 1.37 - 1.38 has only one solution, the origin. It is reasonable that there exists a minimum saving ratio level able to sustain a satisfactory evolution of the economy. Above s_{min} the increasing of s has a twofold beneficial effect: the upper level equilibrium rises and the poverty trap threshold goes slowly downwards. In panel (b), $k^{eq} = k^{eq}(u)$ and $k_{th} = k_{th}(u)$ are on the stage. The evolution of $k^{eq}(u)$

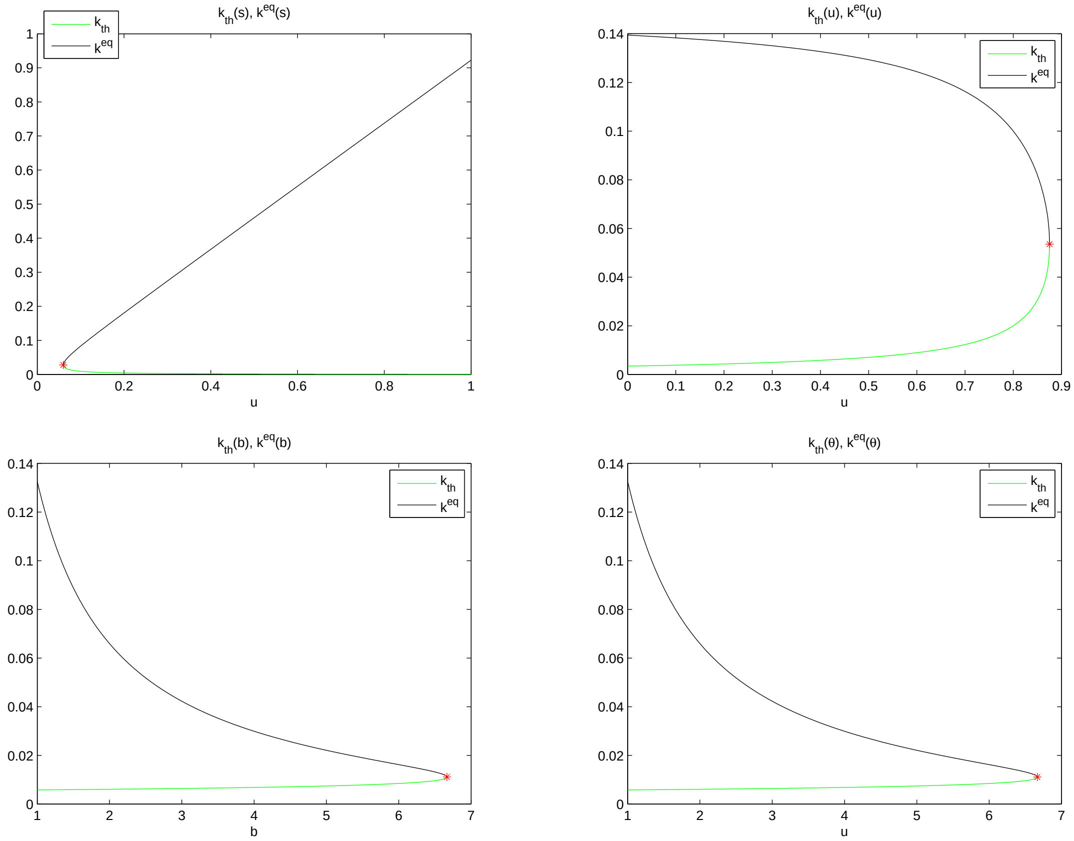

Figure 1.10: Comparative statics examples

and $k_{th}(u)$ follows opposite directions, in black and green respectively: the former decreases, while the latter increases, initially both slowly. Then they rapidly accelerate to the point in which they meet, roughly half way at $u = u_{max}$, and annihilate one another. In other words, as for the performance of the economy, it is initially detrimental putting more effort in the abatement activities, but still acceptable. Then this situation abruptly deteriorate and eventually the system 1.37 - 1.38 looses two equilibria, remaining with only one left, the origin. In panel (c), the story about $k^{eq} = k^{eq}(b)$ and $k_{th} = k_{th}(b)$ is descripted. As b increases the upper equilibrium decreases while the poverty trap level increases. The performance of the economy decreases steeply with respect to the low incresing in the poverty trap threshold. In the context of the linear damage function embedded in our model, this picture suggests that the increasing weight of the production externality affects the long run results of the economy way more than the initial capital condition necessary for the economy to start running in the first place. Panel (d) is almost equal to panel (c), but nevertheless it has something different to tell. While panel (c) is fruitfully red from the left to the right, the ideas conveyed by the dependence of k_{th} and k^{eq} on θ, that is $k_{th} = k_{th}(\theta)$ and $k^{eq} = k^{eq}(\theta)$, are more cutting the other way round: an improving in technologies adoptable against the environamental inefficiency factor obtains much more results in increasing the economic preformance for economies already above their poverty trap threshold than in reducing the poverty trap threshold itself.

1.6 Conclusion

In this paper we studied how pollution, via a damage function, affects the long run performances of the economy. We treated three structurally different economies, with AK, Cobb-Douglas and S-shaped production technology. The introduction of pollution takes away the endogeneity from the Ak model, but it gives dynamics to it. In particular the economy and the enviroment (with pollution as a proxy) have a unique non-trivial equilibrium whose stability has been demonstrated. In the Cobb-Douglas scenario, the strict concavity of the production technology complicates the general analysis, but doesn't preclude the possibility to prove that the unique non-trivial equilibrium of the relative environmental economic model is stable. Things are constitutively different for the S-shaped technology case. Here the origin has its own basin of attraction and so it is not a trivial equilibrium anymore. The origin is stable and all the economy-environment initial conditions that do not overcome the green poverty trap threshold, are eventually condemned to collapse. On the contrary, initial economy-environment that escape the green poverty trap reach the sustainable outcome. A comparitive statics analysis was made for the relevant parameters. Their marginal variations give rise to trade-off between the performance of the economy and the environmental care, except for the case of the environmental inefficiency factor, whose decrease is beneficial both for the economy and for the environment. In the S-shaped stage, the parameters generate bifurcations in the (k,p) phase plane: parameters limit values exist for the economy to have a sustainable steady state. In particular we have stressed the role of the key parameter of the damage function, whose increase can draw some otherwise promising economy to the bottom of a poverty trap.

1.7 Appendix A: Proof of Proposition 1

Existence

It is easy to show that the equilibria of the system 1.3 - 1.4 are:

$$\begin{aligned} k_{eq} &= 0. \\ p_{eq} &= 0. \end{aligned}$$

$$\begin{aligned} k^{eq} &= \frac{(\alpha\, s\, [1-u]^{\epsilon} - \delta_k)\delta_p}{\theta b \alpha [1-u]\delta_k}. \\ p^{eq} &= \frac{\alpha\, s\, [1-u]^{\epsilon} - \delta_k}{b\delta_k}. \end{aligned}$$

We notice that, for the solutions to make sense, it has to be that $(k^{eq}, p^{eq}) > (0,0)$: negative capital and/or pollution are neither considered nor defined. So condition 1.5 on the parameters, that is

$$\alpha\, s\, [1-u]^{\epsilon} - \delta_k > 0$$

is necessary.

Stability

Only the trajectories starting on the k-axis are attracted to the origin, $(k_{eq}, p_{eq}) = (0, 0)$: if $k(0) = 0$, equation 1.3 has no dynamics, while equation 1.4 has just the decay term. Hence the origin can be considered as unstable, having only the locus $(0, p)$ as its region of attraction. The Jacobian matrix associated to (k^{eq}, p^{eq}) is

$$\begin{bmatrix} 0 & -\frac{\delta_p\, \delta k\, (\alpha\, s\, [1-u]^{\epsilon} - \delta_k)}{s\theta\, \alpha^2 [1-u]^{1+\epsilon}} \\ \theta\, [1-u]\alpha & -\delta_p \end{bmatrix}$$

The Determinant D_{AK}, the Trace T_{AK} and the discriminant Δ_{AK} of the charachteristic equation are respectively:

$$D_{AK} = \frac{\delta_p\, \delta_k\, (\alpha\, s\, [1-u]^{\epsilon} - \delta_k)}{\alpha\, s[1-u]^{\epsilon}}, \tag{1.55}$$

$$T_{AK} = -\delta_p \tag{1.56}$$

$$\Delta_{AK} = -4\, \frac{\delta_p\, \delta_k\, (\alpha\, s\, [1-u]^{\epsilon} - \delta_k)}{\alpha\, s[1-u]^{\epsilon}}. \tag{1.57}$$

Under the condition of existence 1.5, the determinant is always positive, the trace is negative and the discriminant of the characteristic polynomial is always negative. It means that the two eigenvalues are complex and conjugate, with strictly negative real part. The upper equilibrium (k^{eq}, p^{eq}) is stable and the trajectories follow spirals to approach the steady state. QED

1.8 Appendix B: Proof of Proposition 2

Existence

The system 1.19 - 1.20 has two equilibria.

$$\begin{aligned} 0 &= \frac{s\,[1-u]^{\epsilon} \alpha k(t)^{\gamma}}{1 + b\, p(t)} - \delta_k k(t). \\ 0 &= \theta\, [1-u]\, \alpha k(t)^{\gamma} - \delta_p p(t). \end{aligned}$$

To obtain the trivial solution $(k_{eq}, p_{eq}) = (0, 0)$, it is possible to proceed by substitution. As for the other solution, let us rearrange the previous system to get:

$$p_k(k) = \frac{s\,[1-u]^{\epsilon} \alpha k^{\gamma-1}}{\delta_k\, b} - \frac{1}{b}. \tag{1.58}$$

$$p_p(k) = \frac{\theta\, [1-u]\, \alpha k^{\gamma}}{\delta_p}. \tag{1.59}$$

The solution lies at the intersection between the two curves $p_k(k)$ and $p_p(k)$. Given that $\gamma \in (0, 1)$, the curve $p_k(k)$ is a continuous, strictly convex and decreasing function whose limits are:

$$\lim_{k \to 0^+} p_k(k) = +\infty. \tag{1.60}$$

$$\lim_{k \to +\infty} p_k(k) = -\frac{1}{b}. \tag{1.61}$$

The function $p_p(k)$ is a continuous, strictly concave and increasing function whose limits are:

$$\lim_{k \to 0^+} p_p(k) = 0. \tag{1.62}$$

$$\lim_{k \to +\infty} p_p(k) = +\infty. \tag{1.63}$$

The difference $d(k) \equiv p_k(k) - p_p(k)$ is a continuous, strictly decreasing function with limits:

$$\lim_{k \to 0^+} d(k) = +\infty. \tag{1.64}$$

$$\lim_{k \to +\infty} d = -\infty. \tag{1.65}$$

It must exist a unique strictly positive point where the function $d(k)$ changes sign: this point is the unique intersection between $p_k(k)$ and $p_p(k)$, and coincides with the equilibrium (k^{eq}, p^{eq}) of the original system 1.19 - 1.20. See, for example, the left panel of Figure 1.6.

Stability

For simplicity we prove Proposition 2 for the case $\delta_k = \delta_p = \delta$. The general case demonstration follows the same ideas, but it is more complicated in terms of algebra. The trivial equilibrium $(k_{eq}, p_{eq}) = (0, 0)$ has the k-axis as the only region of attraction. It can be considered as unstable. As for (k^{eq}, p^{eq}), we know that this solution can be identified with the intersection of the two curves $p_k(k)$ and $p_p(k)$ specified in 1.58 - 1.59. Proceding in the attempt to find an anlytical expression for the equilibrium (k^{eq}, p^{eq}), we equate $p_k(k)$ to $p_p(k)$ obtaining:

$$\frac{s\,[1-u]^{\epsilon}\alpha k^{\gamma-1}}{\delta\, b} - \frac{1}{b} - \frac{\theta\,[1-u]\,\alpha k^{\gamma}}{\delta} = 0.$$

Rearranging the terms of the previous equation we end up with a polynomial in k with fractional powers, whose positive root is k^{eq}:

$$k^{eq} = \mathit{Root\ Of}\ \left(-\alpha\, b\, \theta\, (1-u)\, k^{\gamma} + \alpha\, s(1-u)^{\epsilon} k^{\gamma-1} - \delta\right). \tag{1.66}$$

The Jacobian matrix in function of k^{eq} is the following:

$$\begin{bmatrix} \frac{\alpha\, k^{-1+\gamma}\gamma\, s(1-u)^{\epsilon}\delta}{b\theta\,(1-u)\alpha\,(k^{eq})^{\gamma}+\delta} - \delta & -\frac{\alpha\,(k^{eq})^{\gamma}s(1-u)^{\epsilon}\delta^2 b}{(b\theta\,(1-u)\alpha\,(k^{eq})^{\gamma}+\delta)^2} \\ \theta\,(1-u)\,\alpha\,(k^{eq})^{-1+\gamma}\gamma & -\delta \end{bmatrix}$$

The determinant D_{CD}, the trace T_{CD} and the discriminant Δ_{CD} of the characteristic equation are respectively:

$$D_{CD} = \frac{\left(b^2\theta^2\,(1-u)^2\,\alpha^2 k^{1+2\,\gamma} - (1-u)^{\epsilon}\,\gamma\,\delta\, s\alpha\, k^{\gamma} + 2\, b\delta\,\theta\,(1-u)\,\alpha\, k^{\gamma+1} + \delta^2 k\right)\delta^2}{(b\theta\,(1-u)\,\alpha\, k^{\gamma}+\delta)^2\, k}, \tag{1.67}$$

$$T_{CD} = \frac{\alpha\, k^{-1+\gamma}\gamma\, s\,(1-u)^{\epsilon}\,\delta}{b\theta\,(1-u)\,\alpha\, k^{\gamma}+\delta} - 2\,\delta, \tag{1.68}$$

$$\Delta_{CD} = -\frac{\alpha^2\delta^2\gamma\, s\left(4\, k^{\gamma}\,(1-u)\, b\theta - (1-u)^{\epsilon}\, k^{\gamma-1}\gamma\, s\right)}{(1-u)^{\epsilon}k^{1+\gamma}\,(b\theta\,(1-u)\,\alpha\, k^{\gamma}+\delta)^2\, k^2}. \tag{1.69}$$

Despite the appearance, we are going to show that $D_{CD} > 0$ and $T_{CD} < 0$, when k^{eq} is obtained from 1.66: no matter the sign of the discriminant, the equilibrium k^{eq} is stable, given that the real part of the eigenvalues is always strictly negative. Now we prove that $T_{CD} < 0$ when $k = k^{eq}$:

$$\frac{\alpha\, k^{-1+\gamma}\gamma\, s\,(1-u)^{\epsilon}\,\delta}{b\theta\,(1-u)\,\alpha\, k^{\gamma}+\delta} - 2\,\delta < 0$$

The denominator in the first term of the previous expression is surely positive, so the inequality can be rewritten as:

$$-2\,\alpha\, b\theta\,(1-u)k^{\gamma} + \alpha\,\gamma\, s(1-u)^{\epsilon}k^{\gamma-1} - 2\,\delta < 0$$

and opportunely rearranged as:

$$-\,\alpha\, b\theta\,(1-u)k^{\gamma} + \alpha\, s(1-u)^{\epsilon}k^{\gamma-1} - \delta - \,\alpha\, b\theta\,(1-u)k^{\gamma} + (\gamma-1)\alpha\, s(1-u)^{\epsilon}k^{\gamma-1}) - \delta < 0 \tag{1.70}$$

We know that if $k = k^{eq}$ is the root of the polynomial in 1.66 it means that

$$-\,\alpha\, b\theta\,(1-u)k^{\gamma} + \alpha\, s(1-u)^{\epsilon}k^{\gamma-1} - \delta = 0 \tag{1.71}$$

Plugging 1.71 in 1.70 we get a shorter expression for the trace:

$$-\,\alpha\, b\theta\,(1-u)k^{\gamma} + (\gamma-1)\alpha\, s(1-u)^{\epsilon}k^{\gamma-1} - \delta < 0 \tag{1.72}$$

The previous expression is negative, because $k = k^{eq} > 0$: it follows that the trace is negative at the equilibrium. Now we rearrange the expression of the determinant in 1.67 to get:

$$D_{CD} = F1\,\left(F2 - (1-u)^{\epsilon}\,\gamma\,\alpha\, sk^{\gamma-1} + 2\, b\theta\,(1-u)\,\alpha\, k^{\gamma} + \delta\right).$$

where the terms $F1$ and $F2$, defined as

$$\begin{aligned} F1 &:= \frac{\delta^3}{\left(b\theta\,(1-u)\,\alpha\,k^\gamma + \delta\right)^2}. \\ F2 &:= \frac{b^2\theta^2\,(1-u)^2\,\alpha^2 k^{2\,\gamma}}{\delta} \end{aligned}$$

are always positive. We need to show that the term in the parenthesis is positive, that is:

$$F2 - (1-u)^\epsilon\,\gamma\,\alpha\,sk^{\gamma-1} + 2\,b\theta\,\,(1-u)\,\alpha\,k^\gamma + \delta > 0.$$

On this purpose it can be rewritten as:

$$\begin{aligned} &F2 + [-\,\alpha\,b\theta\,(1-u)k^\gamma + \alpha\,s(1-u)^\epsilon k^{\gamma-1} - \delta] + ... \\ &(1-u)^\epsilon\,(1-\gamma)\,\alpha\,sk^{\gamma-1} + b\theta\,\,(1-u)\,\alpha\,k^\gamma + 2\,\delta > 0. \end{aligned}$$

The term in the in squared bracket is null, as it is immediate from equation 1.71, and the other three terms are all positive. Ergo the determinant of the Jacobian matrix is positive. QED

1.9 Appendix C: Proof of Proposition 3

Existence

We start by recalling the system that describes the equilibria of our model under the assumption of S-shaped technology:

$$0 = \frac{s[1-u]^\epsilon \alpha k(t)^2}{(1+bp(t))(1+\alpha k(t)^2)} - \delta_k k(t). \tag{1.73}$$

$$0 = \frac{\theta[1-u]\alpha k(t)^2}{1+\alpha k(t)^2} - \delta_p p(t). \tag{1.74}$$

For simplicity we prove Proposition 3 for the case $\delta_k = \delta_p = \delta$. The general case demonstration is almost the same, but its albebraic procedure gets more complicated . First we have to notice that the origin is an equilibrium: $(k_{eq}, p_{eq}) = (0, 0)$. Substituting p from the second equation into the first and going through some algebra we get:

$$0 = \alpha\,[b\,\theta(1-u) + \delta]\,k^2 - s\,(1-u)\,\alpha\,k + \delta. \tag{1.75}$$

$$p = \frac{\theta[1-u]\alpha k(t)^2}{(1+\alpha k(t)^2)\delta_p}. \tag{1.76}$$

Equation 1.75 has two real and positive solution as long as the discriminant Δ_{Ss} is greater than zero. Indeed the coefficients of the equation have two changes in sign (Cartesius rule). These two solutions are:

$$k_{th} = \frac{1}{2}\frac{s\,(1-u)\,\alpha + \sqrt{s^2\,(1-u)^2\,\alpha^2 - 4\,\delta\,\alpha\,[b\theta\,(1-u)+\delta]}}{\alpha\,[b\theta\,(1-u)+\delta]}. \quad (1.77)$$

$$p_{th} = \frac{\theta\,(1-u)\,\alpha\,(k_{th})^2}{\left(\alpha\,(k_{th})^2+1\right)\delta}. \quad (1.78)$$

$$k^{eq} = \frac{1}{2}\frac{s\,(1-u)\,\alpha + \sqrt{s^2\,(1-u)^2\,\alpha^2 - 4\,\delta\,\alpha\,[b\theta\,(1-u)+\delta]}}{\alpha\,[b\theta\,(1-u)+\delta]}. \quad (1.79)$$

$$p^{eq} = \frac{\theta\,(1-u)\,\alpha\,(k^{eq})^2}{\left(\alpha\,(k^{eq})^2+1\right)\delta}. \quad (1.80)$$

Summing up, if the discriminant is greater than zero, that is if

$$\Delta_{Ss} = s^2\,(1-u)^2\,\alpha^2 - 4\,\delta\,\alpha\,[b\theta\,(1-u)+\delta] > 0. \quad (1.81)$$

then the solutions are both real and positive.

Stability

The Jacobian matrix of the system 1.37 - 1.38, before the evaluation of the critical points is:

$$\begin{bmatrix} 2\,\frac{\alpha\,ksu}{(\alpha\,k^2+1)(bp+1)} - 2\,\frac{\alpha^2k^3su}{(\alpha\,k^2+1)^2(bp+1)} - \delta & -\frac{\alpha\,k^2sub}{(\alpha\,k^2+1)(bp+1)^2} \\ 2\,\frac{\theta\,u\alpha\,k}{\alpha\,k^2+1} - 2\,\frac{\theta\,u\alpha^2k^3}{(\alpha\,k^2+1)^2} & -\delta \end{bmatrix}$$

The origin is at the border of the domain, but given that the functions involved are sufficiently regular around it, we can still plug $(k=0, p=0)$ in the previous matrix to get that the origin is a stable equilibrium. Indeed the Jacobian matrix evaluated in (k_{eq}, p_{eq})=(0,0) gives:

$$\begin{bmatrix} -\delta & 0 \\ 0 & -\delta \end{bmatrix}$$

Now we are going to demonstrate that

1) The determinant and the trace of the Jacobian matrix associated to (k_{th}, p_{th}) are negative and positive respectively: no matter the sign of the discriminant of the characteristic equation, the equilibrium (k_{th}, p_{th}) is unstable

2) The determinant and the trace of the Jacobian matrix associated to (k^{eq}, p^{eq}) are positive and negative respectively: no matter the sign of the discriminant of the characteristic equation, the equilibrium (k^{eq}, p^{eq}) is stable

For reasons of readability, we report only the final stage of a number of algebraic manipulation performed with Maple 18. The complete sequence of commented Maple is available from the authors upon request.

We start considering (k_{th}, p_{th}). Plugging it in the general Jacobian matrix, we can study the equilibrium properties of this point. The decisive factors to be able to discern the signs of the determinant D_{th}^{Ss} and the trace T_{th}^{Ss} are:

$$D_{th}^{Ss} \propto \alpha\, s\,(1-u)\left(\alpha\, s^2u^2 - 4\,\delta\,[b\theta\,(1-u)+\delta]\right) - \sqrt{\Delta_{Ss}}\left(\alpha\, s^2u^2 - 2\,\delta\,[b\theta\,(1-u)+\delta]\right), \tag{1.82}$$

$$T_{th}^{Ss} \propto -\{\alpha\, s\,(1-u)\left(\alpha\, s^2u^2 - 3\,\delta\,[b\theta\,(1-u)+\delta]\right) - \sqrt{\Delta_{Ss}}\left(\alpha\, s^2u^2 - \delta\,[b\theta\,(1-u)+\delta]\right)\}. \tag{1.83}$$

We should show that in equation 1.82 the following happens:

$$\sqrt{\Delta_{Ss}}\left(\alpha\, s^2u^2 - 2\,\delta\,[b\theta\,(1-u)+\delta]\right) > \alpha\, s\,(1-u)\left(\alpha\, s^2u^2 - 4\,\delta\,[b\theta\,(1-u)+\delta]\right).$$

Observing that

$$\sqrt{\Delta_{Ss}} = \alpha\left(\alpha\, s^2u^2 - 4\,\delta\,[b\theta\,(1-u)+\delta]\right).$$

we proceed dividing both member of the equation by $\sqrt{\Delta_{Ss}}$. We get the following semplification:

$$\alpha\, s^2u^2 - 2\,\delta\,[b\theta\,(1-u)+\delta] > s\,(1-u)\,\sqrt{\Delta_{Ss}}.$$

Now we divide both members by $\alpha\, s^2(1-u)^2$. After some simple algebra we get:

$$1 - \frac{2\,\delta\,[b\theta\,(1-u)+\delta]}{\alpha\, s^2(1-u)^2} > \sqrt{1 - \frac{4\,\delta\,[b\theta\,(1-u)+\delta]}{\alpha\, s^2(1-u)^2}}.$$

Now we do following scaling:

$$t = \frac{\delta\,[b\theta\,(1-u)+\delta]}{\alpha\, s^2(1-u)^2}.$$

The previous inequality becomes:

$$1 - 2t > \sqrt{1-4t}.$$

Taking the square of both members it is easy to show that the last inequality is true whenever $0 < t < 1/4$. Reversing the substitution we get:

$$0 < \frac{\delta\,[b\theta\,(1-u)+\delta]}{\alpha\, s^2(1-u)^2} < \frac{1}{4}. \tag{1.84}$$

Neglecting the trivial case, the inequality in 1.84 is exactly the condition of existence of the solutions. So if the solutions exist, then the determinant of the Jacobian matrix associated to the equilibrium (k_{th}, p_{th}) is negative. Now, adding to both sides of 1.82 the term $\delta\,[b\theta\;(1-u)+\delta]$ yields the relation in 1.83, except for the sign. This means that if the determinant is negative, the trace is positive and the equilibrium (k^{eq}, p^{eq}) is unstable, no matter if the imaginary part of the eigenvalues is different from zero or not.

Now it is the turn of (k^{eq}, p^{eq}). Plugging it in the general Jacobian matrix, we can study the equilibrium properties of this point. The decisive factors to be able to discern the signs of the determinant D^{eq}_{Ss} and the trace T^{eq}_{Ss} are:

$$D^{eq}_{Ss} \propto \alpha\, s\,(1-u)\left(\alpha\, s^2u^2 - 4\,\delta\,[b\theta\;(1-u)+\delta]\right) + \sqrt{\Delta_{Ss}}\left(\alpha\, s^2u^2 - 2\,\delta\,[b\theta\;(1-u)+\delta]\right), \tag{1.85}$$

$$T^{eq}_{Ss} \propto -\{\alpha\, s\,(1-u)\left(\alpha\, s^2u^2 - 3\,\delta\,[b\theta\;(1-u)+\delta]\right) + \sqrt{\Delta_{Ss}}\left(\alpha\, s^2u^2 - \delta\,[b\theta\;(1-u)+\delta]\right)\}. \tag{1.86}$$

It easy to see that the determinant is positive. Indeed, the term $\alpha\, s\,(1-u)\left(\alpha\, s^2(1-u)^2 - 4\,\delta\,[b\theta\;(1-u)+\delta]\right)$, on the left-hand side of relation 1.85, is positive because coincides with $(1-u)\sqrt{D^{eq}_{Ss}}$. While the term $\left(\alpha\, s^2u^2 - 2\,\delta\,[b\theta\;(1-u)+\delta]\right)$, that multiplies $\sqrt{\Delta_{Ss}}$ on the right-hand side, is positive because coincides with $\Delta_{Ss}+2\,\delta\,[b\theta\;(1-u)+\delta]$. In conclusion the determinant of the Jacobian matrix associated with the equilibrium (k^{eq}, p^{eq}) is positive. The trace is negative, thanks to the fact that, except for the sign, the relation in 1.86 can be obtained from 1.85 by adding to both members the quantity $\delta\,[b\theta\;(1-u)+\delta]$. The equilibrium is stable no matter the value of the imaginary part of the eigenvalues, be it zero or different from zero. QED

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Chapter 2

Pollution Diffusion and Abatement Activities across Space and over Time

2.1 Introduction

Understanding the mutual implications between economic activities and environmental degradation has been an important research topic for the last decades (Solow, 1974; Stokey, 1998). Nowadays also policymakers seem to agree that some concrete effort is needed to effectively promote sustainable development (UNEP, 2012), thus the question has become even more relevant than ever. Much attention in literature is placed on the joint evolution of capital and pollution, looking for conditions ensuring that sustainable growth is actually possible (see Xepapadeas, 2005, for a survey; or, among others, Gradus and Smulders, 1993; Bovenberg and Smulders, 1995; Brock and Taylor, 2010). To the best of our knowledge, all the existing works focus only on the temporal economic and environmental dynamics without considering their spatial interaction. This is clearly a strong limitation since pollution is a global phenomenon with no geographical barriers and not restricted within national borders. Moreover, since the typical macroeconomic framework is constructed around a representative agent, it is not possible to analyze how the behavior of certain individuals might affect the behavior of others located in different venues. In order to more realistically describe the world's environmental problems, it is then essential to extend the analysis to a spatial dimension. This is the goal of this paper which thus wishes to shed some light on the joint spatio-temporal evolution of capital and pollution. Since no other paper tries to analyze such a complex and delicate issue, it seems convenient to focus on the simplest possible framework. Specifically, we first consider a Solow-type (1956) model, where agents' decisions are exogenously given. However, the choices of agents in different locations affect one another through a certain diffusion term. Our approach is similar to Brock and Taylor's (2010) in their celebrated green Solow model. With respect to them in order to maintain the model simple we do not allow for (cleaning) technological progress, but we assume that a certain amount of resources is devoted to pollution abatement activities. This means that the saving and environmental protection behavior is exogenous, and agents' spatial heterogeneity is crucial in determining capital and pollution accumulation.

After analyzing such a Solow-type framework, we consider a Ramsey (1928) version of the problem in which economic and environmental policies (i.e., consumption and abatement) are endogenously determined.

Our work tries to combine the macroeconomic and economic geography literature with the growth and environment literature. The growth and environment literature is quite dated and wide (see Smulders, 1999, for a survey), and it mainly shows that in the one-sector framework growth and environment may be compatible or not according to the relative size of the crowding out and productivity effects (Smulders, 1999). The former effect refers to the amount of resources that needs to be subtracted from capital investment in order to protect the environment, while the latter to the potential effects that the environment may induce on production. The macroeconomic and economic geography literature is instead limited and recent. After the seminal works of Krugman (1991, 1992) analyzing spatial dynamic models to highlight the importance of economic geography[1], only recently economic geography has been introduced in canonical macroeconomic models. Camacho and Zou (2004) extend the classical Solow model to the spatial dimension to allow some heterogeneity across different locations, showing the existence of and convergence to a stationary solution. Brito (2004) and Boucekkine et al. (2009) deal with a similar spatial extension of the classical Ramsey model, by focusing on the problems that may arise in a framework of optimal control when the state equations are parabolic differential equations[2]. While all the previous works adopt a neoclassical Cobb-Douglas production technology, Boucekkine et al. (2013b) consider an AK framework allowing to derive an analytical expression for the optimal capital dynamics showing that the spatial structure implies convergence in the capital level across locations even if the returns to capital are constant. Our work is also partly related to another line of research which analyzes the spatial implications and the eventual emergence of patterns and agglomerations in environmental and ecological economics models. Brock and Xepapadeas (2008, 2010) characterize under which conditions diffusion can generate or destroy spatial heterogeneity and agglomeration, with or without the presence of an optimizing planner. Camacho and Pérez–Barahona (2015) study of optimal land use in the presence of local and global pollution, where land is immobile by nature but local actions affect the whole space since pollution flows across locations.

This paper borrows from all these literatures by analyzing the growth and pollution nexus in a context of spatial heterogeneity in which both capital and pollution diffuse across space and accumulate over time. Specifically, the spatio-temporal dynamics of capital is similar to what considered in the macroeconomics and economic geography literature (Boucekkine et al., 2009), while the spatio-temporal dynamics of pollution is consistent with the spatial environmental literature (Camacho and Pérez–Barahona, 2015); the mutual interaction between economic and abatement activities and pollution is enriched according to the feedback externality considered in the growth and environment literature (Smulders, 1999). Indeed, we consider a spatial model where agents are located across a linear city, and economic production has the side effect of generating emissions which increase the stock of pollution, and pollution represents a negative production externality, which therefore reduces capital accumulation. Thus crowding out capital investments will not only reduce pollution but will also allow to alleviate its negative impact on output, due to a

[1]The economic geography literature studies the location, distribution and spatial organization of economic activities across the real world. In particular, it finds out that economic activity is strongly concentrated in a small proportion of the planet's surface, and such a concentration exists not only at world level, but also at several other levels: on metropoli or coasts, within countries, or on particular locations, for many industries (see Fujita et al., 1999; and Fujita and Thisse, 2002).

[2]On the control of partial differential equations arising in the economic growth framework, see also Boucekkine et al. (2013a)

negative productivity effect. We assume that green activities are financed by the tax revenue (capital investments are subject to taxation) and these abatement efforts allow to reduce a certain share of locally generated emissions. The dynamics of both capital and pollution is not trivial since pollution generated in a certain location will affect also the productivity in other locations, via both diffusive and non diffusive mechanisms.

The paper proceeds as follows. Section 2.2 introduces our spatial Solow-type model, summarized by two partial differential equations describing the capital and pollution diffusion, respectively. Since closed form solutions cannot be derived, in section 2.3 we explore the model's outcome through numerical simulations under two alternative scenarios; the first case considers the traditional concave Cobb-Douglas production function while the second one a convex-concave technology in order to represent the growth and environment problems in developed and developing countries, respectively (see Skiba, 1978). We compare the model's outcome under these two different production technology specifications and under different intensities of the diffusion parameters, showing that higher degrees of diffusion can dampen the convergence effects implied by diminishing marginal returns in capital accumulation. Moreover, we show that in the convex-concave production framework the spatial implications of capital and pollution, thanks to two different channels (namely, diffusion and pollution externality), might allow poor regions to escape their poverty traps (or alternatively, it might condemn also rich regions to collapse in the long run). Section 2.4 presents the Ramsey (1928) version of our problem, where the social planner optimally determines consumption and the environmental policy instrument, thus the economy is summarized by an optimal control problem in which the state equations are two partial differential equations. We compare such an optimally planned outcome with what discussed in the previous section highlighting the effects of (optimal) policymaking on the economic and environmental outcomes. Section 2.5 as usual concludes and proposes directions for future research. Technical details about the algorithm used to perform our numerical simulations are discussed in appendix 2.6.

2.2 The Solow-Type Model

We introduce a spatial component in the Solow model, following the approach in Brito (2004), Camacho and Zou (2004), and Camacho et al. (2008). In particular, we assume a continuous space structure to represent that the economy develops along a linear city (see Hotelling, 1929), where not only capital diffuses across different locations but also pollution, even if generated in a specific location, diffuses over the whole economy (Camacho and Pérez–Barahona, 2015). Therefore $k(x,t)$ and $p(x,t)$ denote respectively the capital stock held by and the pollution stock faced by a representative household located in the position x at date t, in a compact interval $[x_a, x_b] \subset \mathbb{R}$, and $t \geq 0$. We abstract from population growth and without loss of generality the population size is normalized to unity. We also assume that the initial capital and pollution distribution, $k(x,0)$ and $p(x,0)$, are known and there is no capital or pollution flow through the boundary of $[x_a, x_b]$ namely the directional derivative is null, $\frac{\partial k(x,t)}{\partial x} = \frac{\partial p(x,t)}{\partial x} = 0$, at $x = x_a$ and $x = x_b$. The spatio-temporal dynamic model is summarized by the following system of partial differential equations:

$$\begin{aligned}
\frac{\partial k(x,t)}{\partial t} &= d_k \frac{\partial^2 k(x,t)}{\partial x^2} + \frac{s(x) f[k(x,t)][1-\tau(x)]}{d[k(x,t)]} - \delta_k k(x,t). \\
&= d_k \frac{\partial^2 k(x,t)}{\partial x^2} + \frac{s(x) f[k(x,t)][1-u(x)]^{\epsilon}}{a + b[p(x,t)]^2} - \delta_k k(x,t). \qquad (2.1) \\
\frac{\partial p(x,t)}{\partial t} &= d_p \frac{\partial^2 p(x,t)}{\partial x^2} + \theta \int_{x_a}^{x_b} [1-u(x')] f[k(x',t)] \varphi(x',x) dx' - \delta_p p(x,t). \qquad (2.2)
\end{aligned}$$

Equation (2.1) describes the evolution of capital. The functions $s(x)$ and $\tau(x)$ describe the savings rate and the environmental tax rate at the location x, respectively; note that they are assumed to be constant over time but heterogenous across space. The production technology $f[k(x,t)]$ uses only capital as an input thus pollution is not a factor of production; however, pollution reduces the level of output through the following damage function $d[k(x,t)] = a + b[p(x,t)]^2$. This specification states that in absence of pollution the production externality is totally irrelevant while as pollution increases output falls more than proportionally. The production technology will be specified as being globally concave or convex-concave later. The local government levies taxes proportional to capital investments to finance environmental protection activities; we assume it wishes to maintain a balanced budget at any point in time, such that the tax revenue is totally devoted to reduce pollution. At location x the tax revenue is $r(x,t) = \tau(x) s f[k(x,t)]$, while abatement activities, $a(x,t)$, decrease a certain share of pollution, $u(x) \in [0,1]$, by employing a certain amount of not consumed output with the following cost $a(x,t) = \mathcal{C}[u(x)] s f[k(x,t)]$, where $\mathcal{C}(\cdot)$ is the cost function ($\mathcal{C}' > 0, \mathcal{C}'' > 0$) of abatement activities which are assumed to be convex and concave as in Kelly (2003). By equating the tax revenue and abatement we obtain that $\tau(x) = \mathcal{C}[u(x)] = 1 - [1-u(x)]^{\epsilon}$ with $\epsilon > 1$, where the cost function is assumed to take the suitable form proposed by Bartz and Kelly (2008). Note that this specification is very convenient since it allows for a one-to-one relationship between the tax rate and the share of abated emissions: if $u(x) = 0$ then $\tau(x) = 0$ such that there is no pollution reduction and (not consumed) production is only allocated to capital accumulation; if $u(x) = 1$ then $\tau(x) = 1$ such that all generated emissions are abated and production is entirely devoted to reduce pollution.

Equation (2.2) illustrates the evolution of pollution. Production generates emissions which increase linearly the stock of pollution and θ measures the degree of environmental inefficiency of economic activities. As in Marsiglio (2015), these abatement activities reduce a share $u(x)$ of emissions, thus $1 - u(x)$ represents unabated emissions. The integral term describes the idea that the emission flows at the position x are due not only to the productive activities at the same spot x but also to the productive activities in different localities, meaning that production at location x', $\forall x' \in [x_a, x_b]$, does impact also on the emission flows faced by location x. Differently from the diffusion term embodied in Laplacian operator $\frac{\partial^2}{\partial x^2}$, the kernel $\varphi(x, x')$ does not describe how pollution spreads across space, but it is a purely static term characterizing proximity externalities not involving mobility; for example, these externalities may be generated by a pollutant that does not diffuse but nevertheless affect the neighbouring areas. In the extreme case we choose the Dirac's delta function as the kernel, the static externality effect would be eliminated (since the integral in the equation for p then evaluates to $f[k(x,t)]$) and the above system would read as:

$$\begin{aligned}\frac{\partial k(x,t)}{\partial t} &= d_k \frac{\partial^2 k(x,t)}{\partial x^2} + \frac{s(x) f[k(x,t)][1-u(x)]^{\epsilon}}{a + b[p(x,t)]^2} - \delta_k k(x,t). \\ \frac{\partial p(x,t)}{\partial t} &= d_p \frac{\partial^2 p(x,t)}{\partial x^2} + \theta[1-u(x)] f[k(x,t)] - \delta_p p(x,t).\end{aligned}$$

With the simultaneous presence of diffusion and static externalities, our model can deal with a composite pollutant that presents both the characteristics connected to diffusion and static externalities. In both equations (2.1) and (2.2), the parameters δ_k and δ_p represent the depreciation rate of capital and pollution, respectively; similarly the parameters d_k and d_p measure the diffusion across space of capital and pollution, respectively. Specifically, $d_k \in [0,1]$ represents the hindrances which dampen spatial diffusion of capital, such as custom barriers, quantitative restrictions or any other government measure with the aim to contain the movements of capital and machines; the term $d_p \in [0,1]$ measures the natural tendency of pollution to spread across space, which to a large extent is out of the control of human beings.

For each fixed pair of savings rate and share of emission abated (s,u), the above system of partial differential equations (2.1) - (2.2) admits a unique solution $(k^{c,u}, p^{c,u})$. The steady state solution is a pair of smooth functions $\bar{k}^{c,u}, \bar{p}^{c,u} \in C^2((x_a, x_b)) \cap C^1([x_a, x_b])$ which satisfy the following system:

$$\begin{aligned} 0 &= d_k \frac{d^2 k(x)}{dx^2} + \frac{s(x) f[k(x)][1-u(x)]^{\epsilon}}{a + bp(x)^2} - \delta_k k(x). \\ 0 &= d_p \frac{d^2 p(x)}{dx^2} + +\theta \int_{x_a}^{x_b} [1-u(x')] f[k(x',t)] \varphi(x',x) dx' - \delta_p p(x).\end{aligned}$$

Let us define $v = (k,p)$ and the following vector-valued function:

$$F(v) := F(k,p) = \begin{bmatrix} \frac{sf(k)[1-u]^{\epsilon}}{a+bp^2} - \delta_k k \\ \int_{x_a}^{x_b} f(k)\varphi dx' - \delta_p p \end{bmatrix}$$

The model (2.1) - (2.2) can be rewritten in vectorial form as an ordinary differential equation in a suitable Banach space as follows:

$$\frac{dv}{dt} = D\Delta v + F(v),$$

where $D = diag(d_i)$ is a diagonal matrix and Δ corresponds to the second order derivative with respect to the spatial variable. Denote by W the real Banach space $C([a,b])$ of continuous vector-valued functions $v : [a,b] \to \mathbb{R}^2$ endowed with the usual norm $\|v\| = \sum_i \sup_{x \in [a,b]} |v_i(x)|$. The problem (2.1) - (2.2) defines a local semiflow Ψ on the Banach space W. For any $v_0 = (k_0, p_0) \in W$, $\Psi(t; v_0)$ provides the unique solution of the problem (2.1) - (2.2) and it satisfies both the maximality and the compactness property (see Mora, 1983). Moreover, it is also possible to prove that the solutions $k(x,t)$ and $p(x,t)$ satisfy:

$$k(x,t) \geq 0,\ p(x,t) \geq 0\ \text{, and } p(x,t) \text{ is bounded from above, } x \in [x_a, x_b],\ t \geq 0.$$

2.3 Spatio-Temporal Dynamics

It is well known that systems similar to ours cannot be solved analytically (with the exception of the linear production function case, as in Boucekkine et al., 2013b). Thus, in order to analyze the behavior of capital and pollution we need to rely on numerical simulations. In the following we consider two different specifications of the production function: a globally concave Cobb-Douglas production function which represents the natural benchmark for our analysis, and a convex-concave production technology, as in Anita et al. (2013). This is introduced in order to allow for the possibility of poverty traps and compare how the outcomes may differ from the canonical Cobb-Douglas case. Such a convex-concave specification rely on Skiba's (1978) argument that decreasing marginal returns can describe only developed economies, while for less developed countries capital may need first to exceed a certain threshold in order for decreasing marginal returns to settle in.

2.3.1 Concave Production: the Benchmark Case

Consider first a neoclassical, globally concave, Cobb-Douglas production function:

$$f[k(x,t)] = Ak(x,t)^q,\ q < 1, \tag{2.3}$$

where A is a productivity parameter and $q < 1$ denotes the capital share. In this case, the semiflow Ψ is monotone and concave, guaranteeing the convergence to a nontrivial steady state. To visualize the long-run behavior of the system (2.1) - (2.2), we perform a numerical simulation, with the following parameter values and initial conditions:

$$\begin{cases} s(x) = 0.2,\ u(x) = 0.5,\ \theta = 0.02,\ \epsilon = 1.5,\ x_a = -1,\ x_b = 1, \\ \delta_k = 0.05,\ \delta_p = 0.05,\ a = 1,\ b = 0.01, \\ k_h = 0.86,\ p_h = 0.86,\ d_h = 0.1,\ \sigma_h^2 = 0.06, \\ k_l = 0.22,\ p_l = 0.22,\ d_l = 0.001,\ \sigma_l^2 = 0.5, \\ A = 100,\ q = 0.33,\ d_k = d_p = d_h, \\ k(x,0) = k(x,0)_h = k_h e^{-\frac{x^2}{\sigma_h^2}},\ p(x,0) = p(x,0)_h = p_h e^{-\frac{x^2}{\sigma_h^2}}, \\ \varphi(x'x) = \frac{1}{\sqrt{2\pi}\varepsilon} e^{-\frac{(x-x')^2}{2\varepsilon^2}},\ \text{with } \varepsilon = \frac{1}{\sqrt{2\pi}}. \end{cases} \tag{2.4}$$

In Section 2.4 we will let the saving rate and share of abated emissions be endogenously determined, but for the time being we simply assume they are constant over time but non-homogeneous across space. We firstly allow the saving rate, s, to be constant at any location, namely we set $s(x,t) = 0.2$ which represents a standard value for Solow-type models (in a spatial growth context, see Camacho and Zou, 2004). The depreciation rate of capital and the capital share, $\delta_k = 0.05$ and $q = 0.33$ respectively, are consistent with most of the growth literature (see Barro and Sala-i-Martin, 2004). The pollution decay rate, δ_p is set equal to 0.05 (see, for example Saltari e Travaglini, 2014). The share of abated emission, $u(x,t)$, is constant at its average value, $u(x,t) = 0.5$. The cost parameter, ϵ, is set equal to 2 for the sake of simplicity: different choices of ϵ around such a value do not change our conclusions. The diffusion coefficients, d_k and d_p, are set arbitrarily to allow the diffusion to clearly affect the economic and environmental outcomes, permitting thus to appreciate the consequences of this force on the economic and environmental variables; specifically, we will consider high and low diffusion coefficients, d_h and d_l respectively. The initial distribution of capital and pollution, $k(x,0)$ and $p(x,0)$, are chosen to mimic an initial non-homogeneous distribution of k and p. The auxiliary parameters k_h, k_l, p_h, p_l, σ_h^2 and σ_l^2, shape the initial profile of capital and pollution, and are set to make easier the comparisons between the different scenarios in the remainder of the paper. The environmental production inefficiency, θ, the total factor productivity, A, and the two damage function parameters, a and b, are set in such a way to improve the readability of our (graphical) results, that is to facilitate an immediate grasp of the main ideas just by a glance at the figures: their numerical values do not affect any of our qualitative results.

It is worth to comment on the specification of the kernel function $\varphi(x', x)$. If our spatial domain was infinite, the function $\varphi(x', x)$ would possess the following property: $\int_{-\infty}^{+\infty} \varphi(x', x)dx' = 1$. That would be interesting because we could think about the Dirac's delta function as the limit distribution of our kernel function. This point of view could then help us to interpret our integro partial differential system as a natural generalization of a standard reaction diffusion system. Our domain is actually finite and the specified kernel function introduces some asymmetries. In particular the closer the position x is to the left (right) boundary x_a (x_b), the more truncated is the left (right) tail of the kernel function, and the less contribution the integral provides to the pollution faced at the point x. Hence the central regions around $x = 0$ are the most affected by the static spatial externalities. In other words, the specified kernel function is chosen to allow the Dirac's delta function to be its limit distribution, and to create an externality with spatial heterogeneity effects, as we will see.

The evolution of both capital and pollution are illustrated in Figure 2.1. Intuitively in our benchmark Cobb-Douglas case, since the production function is concave, the economy does not have to face take off problems[3] and therefore as time elapses, both capital and pollution reach their positive steady state levels. Therefore, the economy develops along a sustainable path: capital and pollution first have dynamics and then stabilize at their positive long run level. Note that the long run steady state is non-homogeneous across space; to understand such an outcome we need to consider the two opposite forces acting on the spatial distribution of capital and pollution. On the one hand, diffusion is a convergence mechanism, that is diffusion tends to smooth the spatial differences out, translating the idea of decreasing returns to capital: capital moves toward regions where the returns are higher, that is regions in which capital is scarce. If there are no spatial exogeneous differences either in the production function or in the saving rate or in the abatement share, namely if f, s and u do not explicitly depend on x, this smoothing out process continues until the initial space dependent profiles $k(x,0)$ and $p(x,0)$ fade away (see Brito, 2004, Camacho and Zou 2004).

[3]See Skiba (1978) and Sachs et al. (2004) for a critical discussion of the assumption of decreasing marginal returns, embedded in a concave production function. They claim that such a formulation can be considered an accurate description only of developed economies, but cannot be applied to less developed economies, often facing poverty traps; for this reason Skiba (1978) suggests that convex-concave production functions can better describe real world economies.

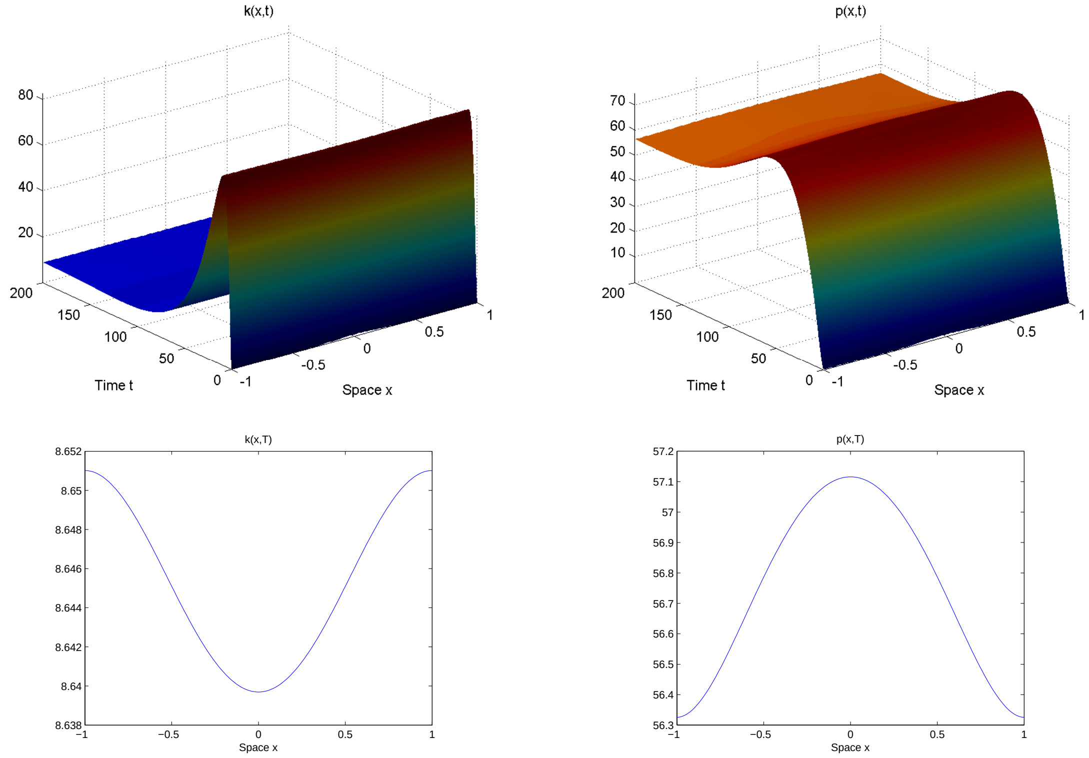

Figure 2.1: Cobb-Douglas production case: evolution of capital and pollution ($d_k = d_p = d_h$ and $k(x,0) = k(x,0)_h$)

On the other hand, the integral term represents a divergence mechanism: it introduces (static) non-diffusive spatial externalities at location x, being the positions around $x = 0$ the most affected venues by the amount of pollution available in the overall economy. We can think about such non-diffusive contributions as the effect of that part of the stock of pollution that is really hard to move, such that it cannot diffuse but still affects the areas surrounding the production site. Looking at the steady state profiles of capital and pollution, at the bottom panels of Figure 2.1, it is possible to conclude that overall the strength of the diffusive force is not enough to completely overcome the effect of the spatial externality. The regions with the highest levels of steady state pollution are the central ones, since the integral-induced effect is stronger. As a consequence, given the role of the damage function, these are even the regions where less capital is accumulated in the long run. In order to more easily visualize the divergence role of the kernel integral we repeat the above simulation with the same parameter values, except for the absence of integral term, that is we set $\phi(x) = \delta(x)$. The system becomes:

$$
\begin{aligned}
\frac{\partial k(x,t)}{\partial t} &= d_1 \frac{\partial^2 k(x,t)}{\partial x^2} + \frac{s(x) f[k(x,t)][1-u(x,t)]^{\epsilon}}{a + bp(x,t)^2} - \delta_k k(x,t). \\
\frac{\partial p(x,t)}{\partial t} &= d_2 \frac{\partial^2 p(x,t)}{\partial x^2} + \theta[1-u(x,t)] f[k(x,t)] - \delta_p p(x,t).
\end{aligned}
$$

As we can see in Figure 2.2, capital and pollution reach a spatially homogeneous steady state, given that now diffusion is the only force taking place. Thus, by comparing the bottom part of Figure 2.1 and Figure 2.2 it is possible to appreciate the contribution of the non-diffusive spatial externalities (i.e., the integral term) in the fully fledged model. Non-diffusive spatial externalities by introducing some spatial heterogeneity in the evolution of pollution allow some

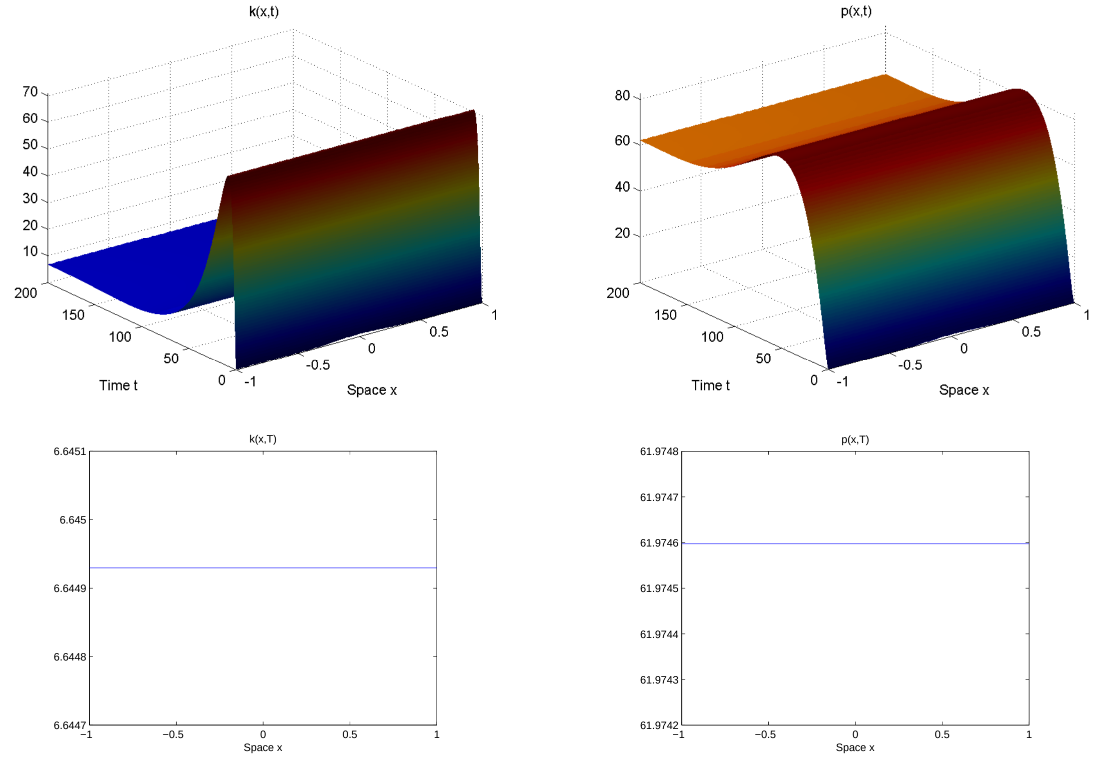

Figure 2.2: Cobb-Douglas production case: evolution of capital and pollution (with $d_1 = d_2 = 1$ and $k_0 = 1$, along with $\phi(x) = \delta(x)$).

spatial heterogeneity to persist in the long run level of both capital and pollution.

2.3.2 Convex-Concave Production and Poverty Traps

In order to enrich both the economic and environmental dynamics and consider the eventual presence and effects of poverty traps on the accumulation and diffusion of capital and pollution over time and across space, we now focus on a convex-concave production function as suggested in Skiba (1978). For the sake of simplicity, as in Anita et al. (2013) we consider the following functional form:

$$f[k(x,t)] = \frac{\alpha_1 k(x,t)^q}{1 + \alpha_2 k(x,t)^q}, \; q > 1, \tag{2.5}$$

where α_1 is a productivity parameter, and α_2 measures eventual diseconomies[4]. Because the production function is S-shaped, the analysis of the model is complicated due to lack of concavity of the evolution operator; some analytical results about local stability of steady states can be found in Capasso and Maddalena (1982). Capasso et. al. (2010) analyze a spatial Solow model under a similar non-concave production function showing the existence of three equilibria, where two are stable and one is unstable, with the unstable equilibrium corresponding to the poverty trap threshold,

[4]Even if it can be discussed whether this specific formulation of a convex-concave production is relevant or can be empirically supported, there are no doubts on the fact that the standard neoclassical globally concave production function is overly simple to give rise to realistic results. The formulation we present here maintains very useful tractable properties and allows for the existence of poverty traps. In particular, if $q > 1$, it satisfies most of the neoclassical properties, since it shows constant returns to scale, positive and diminishing (only for higher capital levels) marginal returns which converge to zero as capital gets larger and larger. If $q \leq 1$, instead the production function results to be concave, and the results will not be qualitatively different from the Cobb-Douglas case. For this reason we consider such a type of production function to be a better benchmark for investigating real world problems.

as discussed in Skiba (1978). By assuming such a specification for the production function our model might give rise to similar poverty trap outcomes. Before looking more in depth to what this implies for our economic and environmental framework, we need to carefully understand what a poverty trap might represent in our spatial framework. In Skiba's (1978) seminal work, since a space structure is completely missing the poverty trap threshold is the value of the initial capital, say k_{th}, that separates the basins of attraction of the lower and the upper stable equilibria: if the economy starts below this threshold, $k_0 < k_{th}$, the system evolves towards the lower equilibrium k_l, while the upper equilibrium k_u is eventually reached if the initial capital is above k_{th}, i.e. $k_0 > k_{th}$; basically, k_{th} is a point in the capital domain representing the unstable equilibrium of the economy. The introduction of a space structure like ours complicates dramatically the analysis. Indeed, since we are dealing with an infinite dimensional problem, the equivalent notion of poverty trap threshold is a $\mathcal{C}^2$-function, representing the unstable (middle) equilibrium of the system: if the initial allocation of capital were exactly equal to this threshold function $k_0(x) = k_{th}(x)$, the system would not evolve at all. All the other initial distributions of capital are attracted either to the low or to the high equilibrium, but it is not possible to order them with an inequality sign, as in the one dimensional case, because there is no such an ordering in the space of functions. Nevertheless, in the absence of diffusion, it is possible to define a *"spatial poverty trap threshold"* function: this spatial threshold will help us to appreciate the effects of diffusion in a number of scenarios. In order to understand what this spatial threshold function may represent, let us neutralize for a while both the diffusion (by setting $d_k = d_p = 0$) and all the spatial exogenous heterogeneities (setting both the saving rate and the share of abated emission equal to some constant value, $s(x) = s$ and $u(x) = u$, and the kernel function equal to the Dirac's delta function, $\phi(x) = \delta(x)$), apart from the initial capital and pollution distribution (letting $k(x, 0) = k_0(x)$ and $p(x, 0) = p_0(x)$); this implies that the space variable x in our system is now de facto a parameter. The system of partial differential equations (2.1) - (2.2) boils down to the following parametric system of ordinary differential equations, which at each point x can be solved along with the initial conditions $k(x, 0) = k(x)$ and $p(x, 0) = p(x)$:

$$\frac{dk_x(t)}{dt} = \frac{sf[k_x(t)][1-u]^\epsilon}{a + bp_x(t)^2} - \delta_k k_x(t). \tag{2.6}$$

$$\frac{dp_x(t)}{dt} = \theta[1-u]f[k_x(t)] - \delta_p p_x(t). \tag{2.7}$$

The above system has clearly three equilibria, two stable and one unstable. The unstable equilibrium k_{th} is the poverty trap threshold, exactly as in Skiba's (1978) framework. Now we can define $k(x) = k_{th}$ as our spatial poverty trap threshold, that is a function representing what the poverty trap threshold might be in a framework with no diffusion.

For the simulations[5] in this section, the values of the parameters and initial conditions are the following:

$$\begin{cases} s(x) = 0.2,\ u(x) = 0.5,\ \theta = 0.02,\ \epsilon = 1.5,\ x_a = -1,\ x_b = 1, \\ \delta_k = 0.05,\ \delta_p = 0.05,\ a = 1,\ b = 0.01,\ \alpha_1 = 1,\ \alpha_2 = 1, \\ k_h = 0.86,\ p_h = 0.86,\ d_h = 0.1,\ \sigma_h^2 = 0.06, \\ k_l = 0.22,\ p_l = 0.22,\ d_l = 0.001,\ \sigma_l^2 = 0.5, \\ A = 100,\ q = 4,\ d_k = d_p = d_h \text{ or } d_l, \\ k(x,0) = k(x,0)_h = k_h e^{-\frac{x^2}{\sigma_h^2}},\ p(x,0) = p(x,0)_h = p_h e^{-\frac{x^2}{\sigma_h^2}} \text{ or} \\ k(x,0) = k(x,0)_l = k_l e^{-\frac{x^2}{\sigma_l^2}},\ p(x,0) = p(x,0)_l = p_l e^{-\frac{x^2}{\sigma_l^2}}, \\ \varphi(x'x) = \frac{1}{\sqrt{2\pi}\varepsilon} e^{-\frac{(x-x')^2}{2\varepsilon^2}},\ \text{with } \varepsilon = \frac{1}{\sqrt{2\pi}}. \end{cases} \tag{2.8}$$

In Figure 2.3 we compare two initial capital allocations (we do leave pollution aside for the time being), their dynamics and their long run behavior, in the absence of diffusion. In the top panels the red dotted line represents the spatial threshold introduced above[6]: we can see that the share of rich (i.e., those lying above the threshold) and poor (i.e., those lying below the threshold) regions is the same in the two scenarios, while the initial spatial distribution of capital is clearly different (in the left panel rich regions are relatively much richer than poor regions, while in the right panel rich and poor regions do not show such a strong difference in their initial capital endowments). In the absence of capital diffusion the only thing that matters for the dynamic evolution of the economy is the initial capital level, thus it turns out that the two scenarios have similar dynamics (middle panels), and identical long run spatial profiles (bottom panels): rich regions converge towards the high equilibrium, while poor regions are trapped in the low (zero capital) equilibrium.

In Figure 2.4 diffusion steps in and the two scenarios have no longer the same dynamics and long run outcome: in the left panels, where the rich regions have a more abundant endowment of capital, the diffusion-induced effect is beneficial and even poor regions are able to reach the high equilibrium. However, diffusion is not beneficial a priori, as we can see from the right panels of the same Figure: if the capital endowment of the rich regions is not sufficiently large, diffusion could even bring every region, both rich and poor regions, to collapse. What this result suggests is that the initial capital distribution plays a fundamental role in determining whether the convergence mechanism triggered by diffusion might have overall a positive or negative effect on the whole spatial economy; in particular, if rich regions are particularly rich diffusion can help poor regions to escape the poverty trap. In economic terms, this means that in a spatial economy with heterogeneous capital endowments, whenever rich regions are rich enough the spatial economy might have internally the resources required to provide poor regions with the big push they need to escape poverty (Sachs et al., 2004); whenever rich regions are not rich enough the spatial economy might fail in the attempt to help poor regions and as a result also rich regions will be pulled to poverty.

In this discussion about the strength of diffusion in shaping the dynamics of the spatial economy, we have not

[5]In the following simulations we set arbitrarily $q = 4$, since the qualitative results are equivalent for any other value of q such that $q > 1$, which represents the relevant case for generating an S-shaped production function, according to (2.5).

[6]In this paper, as a matter of expositional simplicity, we refer to the set of locations lying above the spatial threshold as "rich regions" and the set of those lying below the spatial threshold as "poor regions". Note that also within the group of rich and poor regions there exists a certain degree of heterogeneity in the initial distribution of capital, what we we will be referring to as "spatial capital distribution".

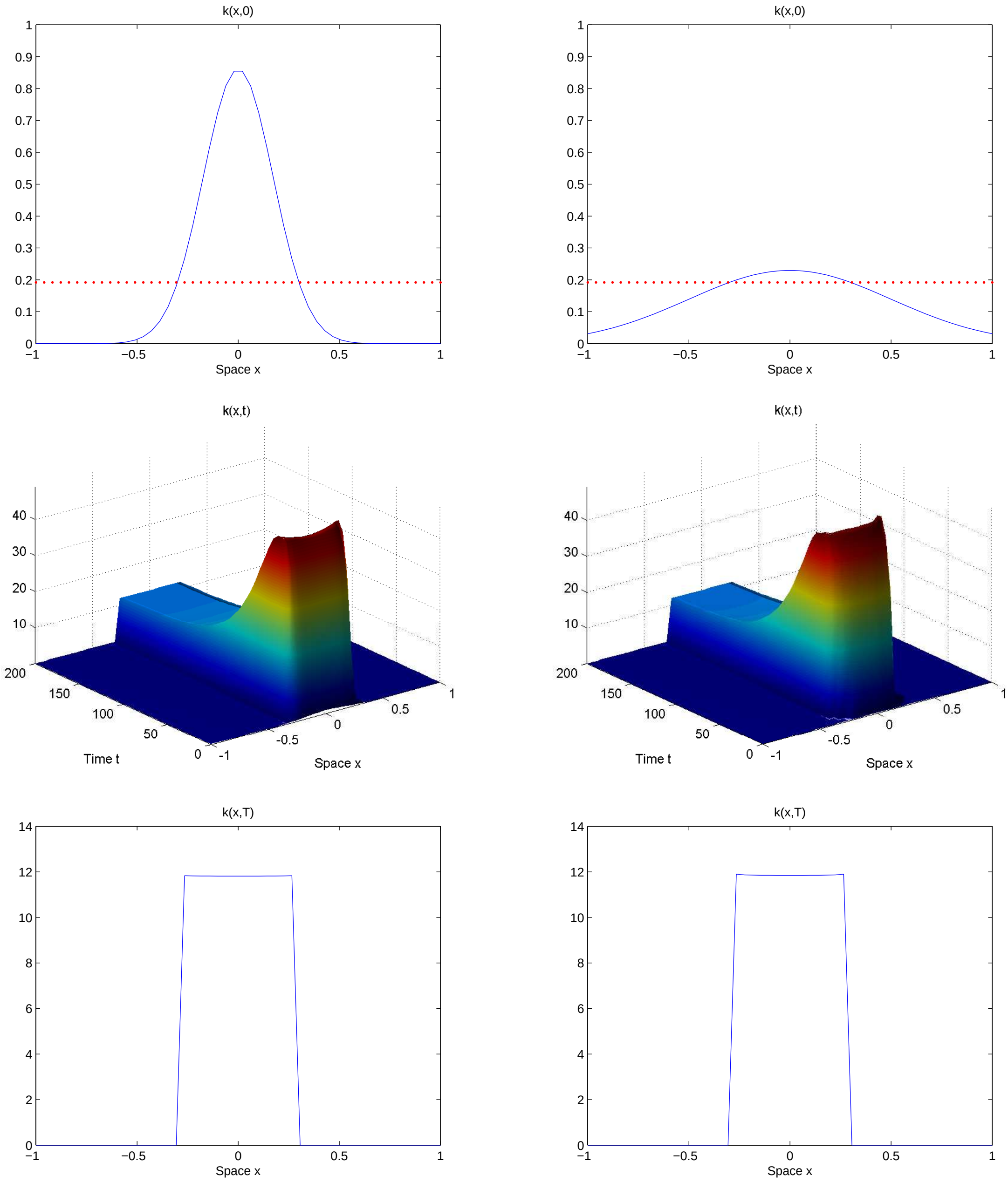

Figure 2.3: S-shaped production case: initial capital allocation for $k(x,0) = k(x,0)_h$ (left panel), and $k(x,0) = k(x,0)_l$ (right panel),spatial poverty trap threshold and evolution of capital and pollution (no diffusion).

considered the effects of pollution yet. However, pollution crucially affects the dynamics of capital through two terms: the degree of environmental inefficiency of economic activities (the parameter θ) and the damage function (the parameters a and b); if any of these terms increases, the spatial threshold rises too while the high equilibrium level of capital falls, making escaping the poverty trap more difficult and at the same time the long run outcome less satisfactory. We can observe such an outcome in Figure 2.5, where both the economy on the left and right panels share the same initial capital endowments as the economy on the left panels of the previous Figure 2.4 (that is, rich regions are substantially rich), but they are characterized by different values of b and θ (we increase b in the economy on the left, and we increase θ in the economy on the right). The dynamics are slightly different but the long run

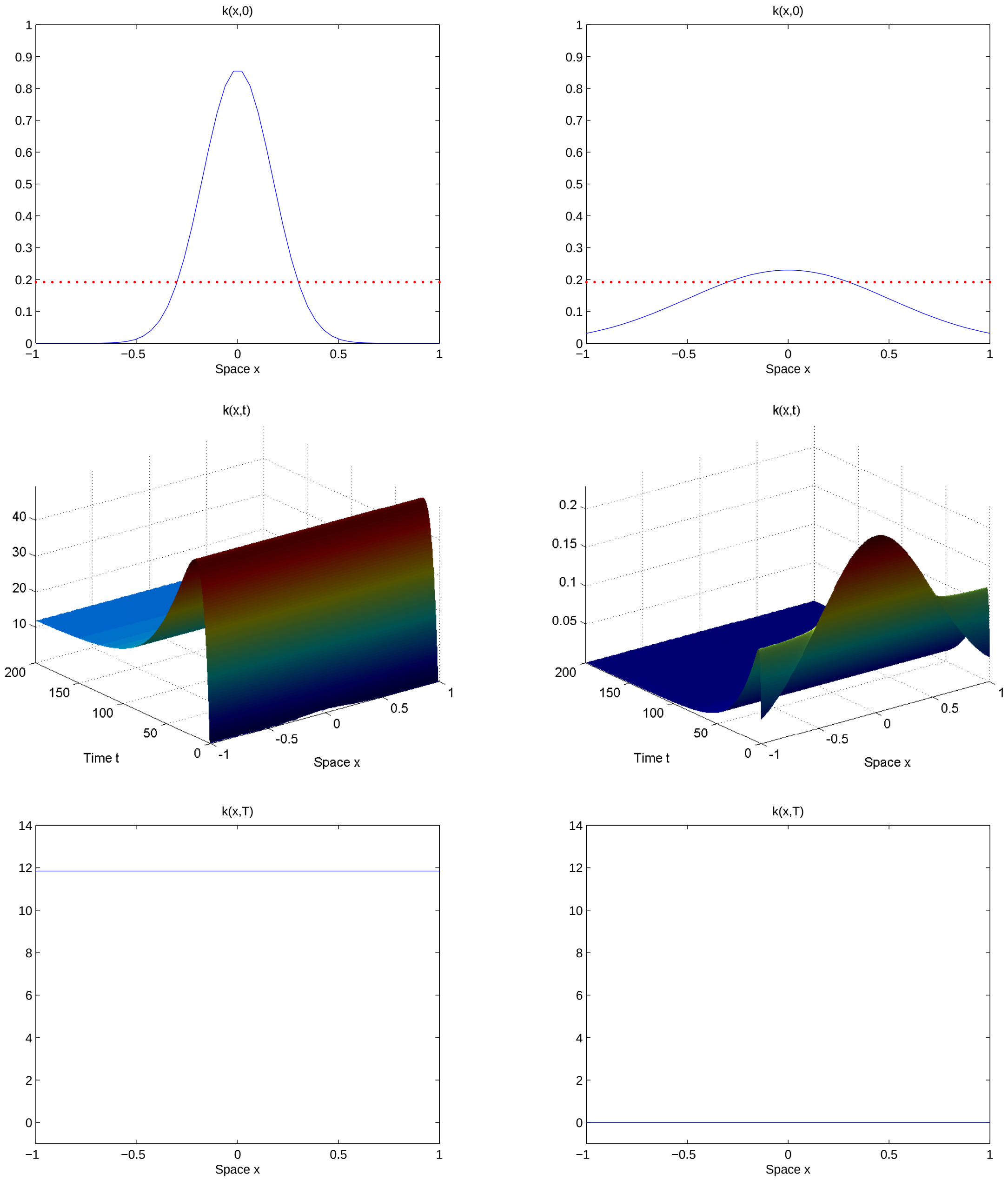

Figure 2.4: S-shaped production case: initial capital allocation for $k(x,0) = k(x,0)_h$ (left panel), and $k(x,0) = k(x,0)_l$ (right panel), spatial poverty trap threshold and evolution of capital and pollution (with diffusion).

behavior of the economies is the same: long run collapse. We can note that the two parameters affect the dynamics of capital in a very different way: when b increases the rich regions in which also the initial pollution is high, immediately experience the stronger negative effects associated with the pollution externality losing their capital advantage, while when θ increases the rich regions keep their initial capital advantage on poor regions for some time. This highlights that the damage function (by affecting the capital dynamic equation) affects the economy even in the short run and its implication on the spatial distribution of capital can be detected soon, while the degree of environmental inefficiency (by affecting capital dynamics only indirectly through the pollution dynamic equation) has only long run effects, even though they are similarly negative. By comparing Figure 2.5 and Figure 2.4 we can note that not only the initial capital distribution matters for determining whether the spatial economy will be able to to converge to the high capital equilibrium level or not, but also the pollution-induced effect does. Indeed, if pollution strongly affects output (directly through the damage function, or indirectly through the degree of environmental inefficiency) even if rich regions are substantially rich the whole spatial economy might be condemned to collapse in the long run.

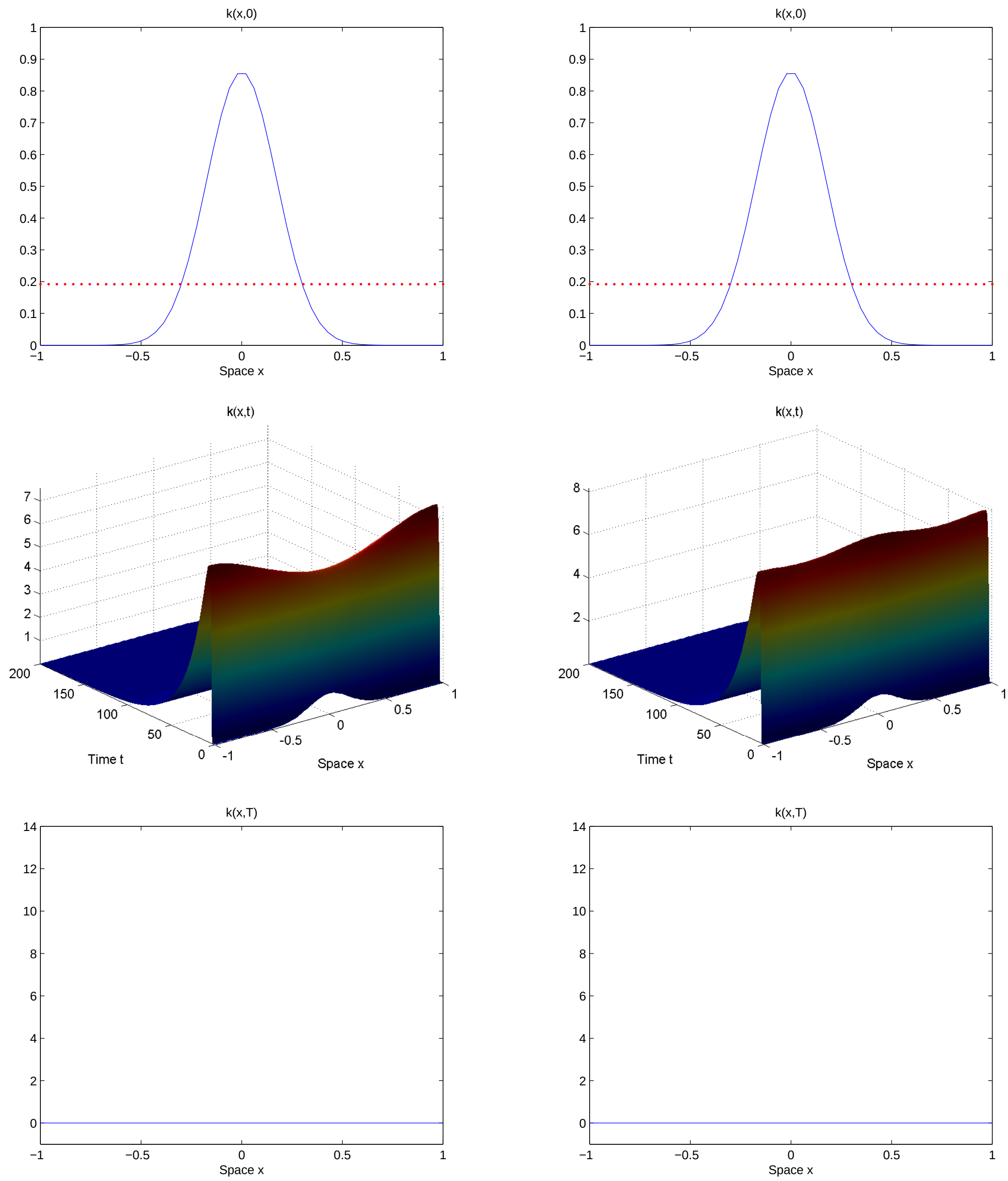

Figure 2.5: S-shaped production case: initial capital allocation for $k(x,0) = k(x,0)_h$ with higher b (left panel), and $k(x,0) = k(x,0)_h$ with higher θ (right panel), spatial poverty trap threshold and evolution of capital and pollution (with diffusion and changes in pollution effects).

We now turn back to the fully fledged model and we explore the behavior of (2.1) - (2.2), subject to (2.5). We consider different values for d and $k(x,0)$ in order to illustrate the effect of the diffusion and the initial capital distribution on the long run outcome of the spatial economy. In particular, we use the values $d = d_h$ and $d = d_l$, which correspond to high diffusion and low diffusion, respectively, for both capital and pollution. We alternatively chose $k(x,0) = k(x,0)_h$ or $k(x,0) = k(x,0)_l$, as in the upper panels of Figure 2.4. As in the previous discussion, the share or rich and poor regions are the same in the different scenarios. The goal of the simulations illustrated from Figure 2.6 to Figure 2.9 is to show the combined effects of diffusion and initial capital distribution on the long run economic and environmental outcome. Figure 2.6 presents the steady state solutions and the solutions surfaces in

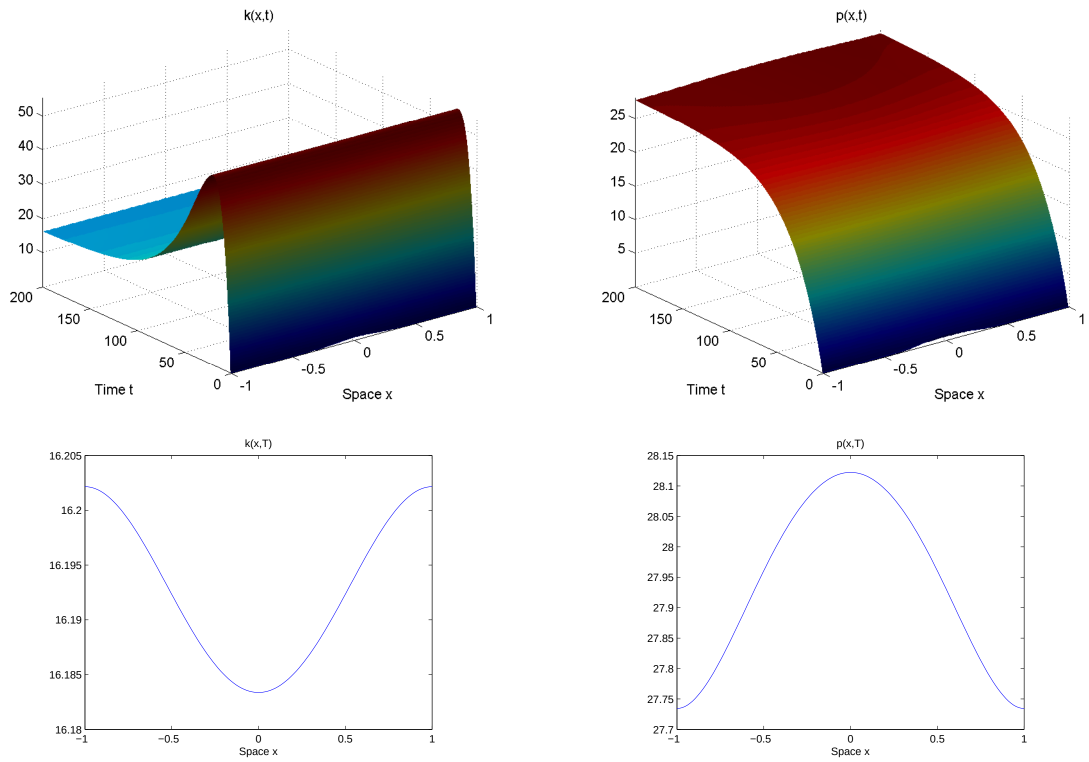

Figure 2.6: S-shaped production case: evolution of capital and pollution (with $k(x,0) = k(x,0)_h$ and $d = d_h$).

the case in which $d = d_h$ and $k(x,0) = k(x,0)_h$. Both k and p approach non-trivial steady states. The economy evolves along a sustainable path along which capital and pollution experience dynamics before reaching their steady states eventually. By observing the final time profile of k and p it is possible to note that a long run heterogeneity in space survives, even if the value of the diffusion coefficient is high. Comparing Figure 2.1 with Figure 2.6, whose simulations rely on the same set of parameters and functions apart from the production function, and considering that the relative peak-bottom distance in their respective final time profiles is almost exactly the same in the two cases, we can conclude that what really matters in generating a spatial heterogeneity is the externality embedded in the kernel function, as it was previously shown for the Cobb-Douglas case. This is a noticeable result because it shows how the effect of non-diffusive pollution externalities shape the long run spatial profile of the economy: rich regions enjoy a sustainable path, but their long run achievements are negatively affected by the grater amount of pollution they eventually suffer. However, even if the diffusion (that is the possibility for capital and pollution to freely flow across regions) does not overcome the exogenous asymmetries introduced by the kernel function, it turns out to have overall a positive effect. Indeed, since the initial capital endowment in rich regions is sufficiently abundant, this permits also poor regions to achieve a sustainable steady state. Next, we reduce the level of capital and pollution diffusion over space by setting $d = d_l$. Figure 2.7 shows that both k and p reach non-trivial, non-homogeneous steady states in

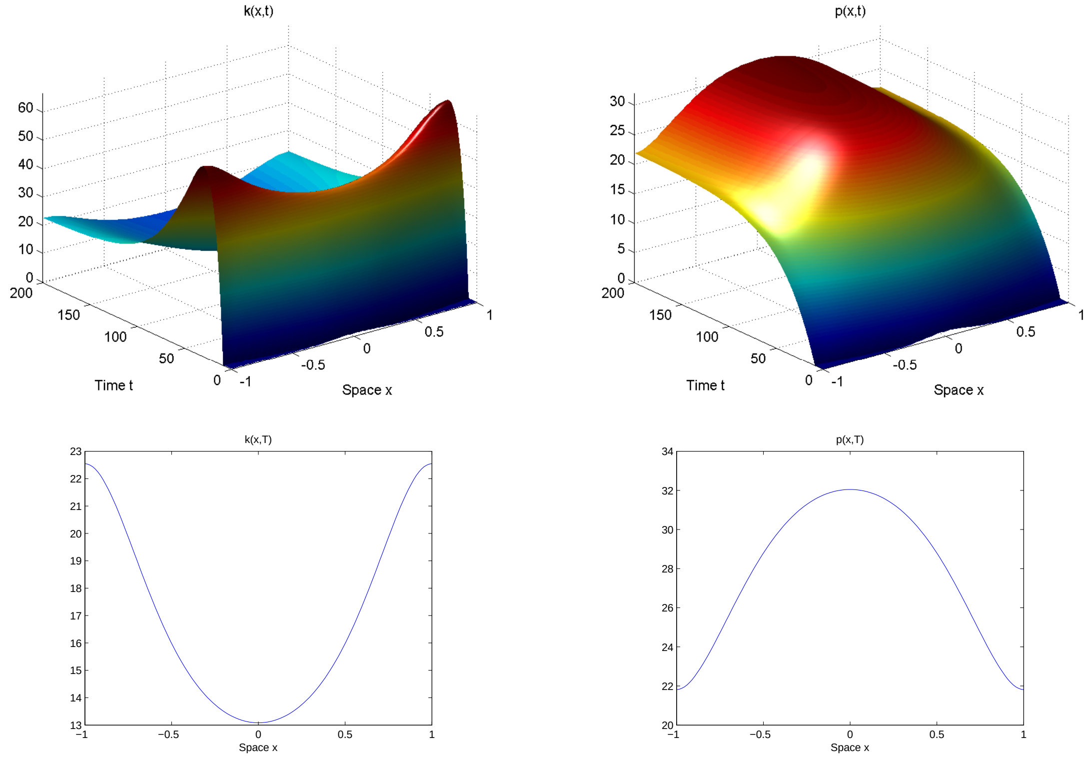

Figure 2.7: S-shaped production case: evolution of capital and pollution (with $k(x,0) = k(x,0)_h$ and $d = d_l$).

this scenario. The spatial heterogeneity is now considerably sharper than before, but nevertheless the initial capital endowment guarantees a sustainable steady state for all the regions in the domain.

Then we reduce the overall initial capital by setting $k(x,0) = k(x,0)_l$. When $d = d_h$, the capital and pollution diffusion levels are high enough to ensure that all regions suffer. Figure 2.8 confirms that in such a case the resulting solutions for both k and p approach zero. Even if in the initial spatial distribution of capital there are some rich regions, diffusion drains away resources so heavily that eventually all the regions, both rich and poor, are doomed to the same dire destiny. Differently from the previous case, where diffusion is actually helping poor regions to escape their poverty trap, here the possibility for capital to freely flow toward underdeveloped regions is quite negative ruling out the possibility of achieving a sustainable path for all the regions in the domain. Note that, as discussed earlier, the initial capital distribution is crucial in this respect.

Not only the initial capital distribution matters but also the intensity of the diffusion parameter does. In fact, if the diffusion terms are smaller, as set equal to $d = d_l$ as in Figure 2.9, both k and p reach a non-trivial, non-homogeneous steady state (exactly the same steady state as in Figure 2.7, being the initial value of k the only difference between the two simulations). Slowing down the diffusion of capital helps every region to grow along a sustainable path. When $k(x,0) = k(x,0)_l$, a variation of the diffusion coefficient from $d = d_l$ to $d = d_h$ is devastating, leading every region in the spatial economy to collapse to a zero capital level. In other words, reducing the diffusion can be beneficial for some initial capital distributions: indeed, while before the reduction of the diffusion, the initial capital distribution of the spatial economy belongs to the basin of attraction of the trivial steady state (Figure 2.8), after the reduction, the same initial capital distribution ends up in the basin of upper steady state level (Figure 2.9). However, note that this

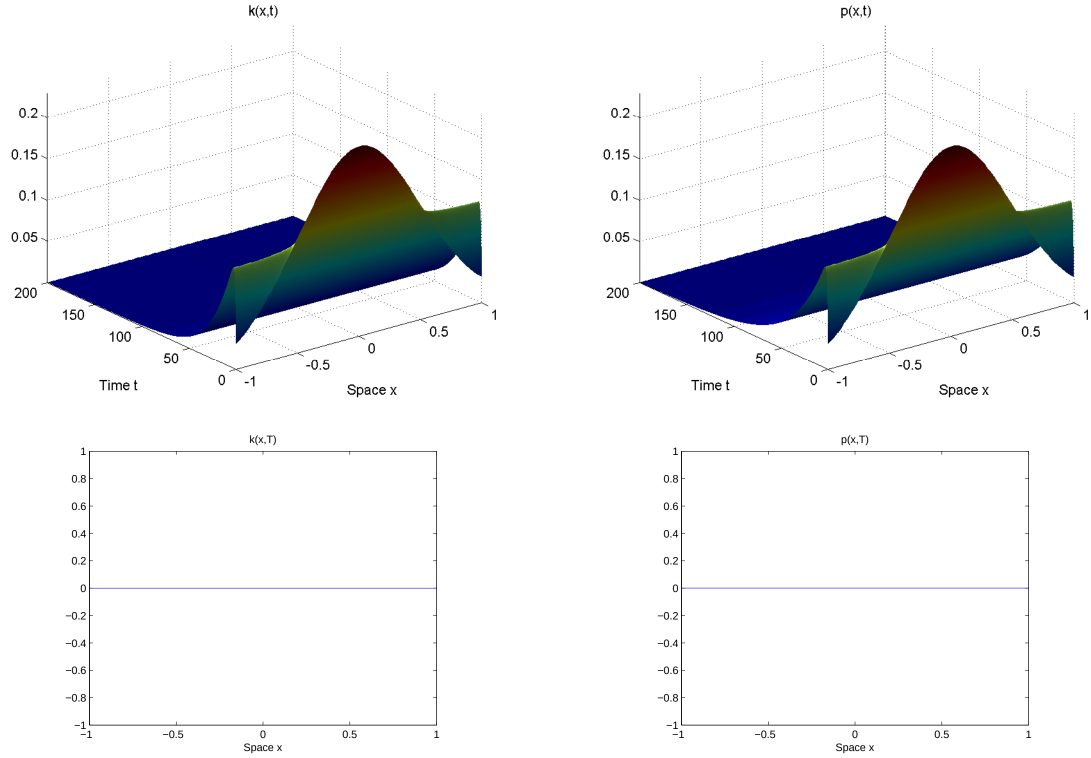

Figure 2.8: S-shaped production case: evolution of capital and pollution (with $k(x,0) = k(x,0)_l$ and $d = d_h$).

reduction of diffusion does not preclude the basin of attraction of the trivial steady state to exist: in this last scenario ($k(x,0) = k(x,0)_l$ and $d = d_l$), a slight reduction in the parameter k_l, for example, can lead the economy to the way to collapse[7]. These numerical results clearly show that in an S-shaped technology framework, it is no longer obvious that the spatial economy develops along a sustainable path[8].

By comparing the results obtained under a neoclassical and S-shaped production function, respectively given in (2.3) and (2.5), we can understand the main implications of poverty traps on capital and pollution. While collapsing paths are a possible outcome under a convex-concave production function (and they crucially depend on both the initial distribution of capital and the intensity of the diffusion parameters), such a possibility is completely ruled out by the concave Cobb-Douglas technology. Indeed, such a formulation implicitly assumes that the critical capital threshold level coincides with the zero capital level and therefore it has already been exceeded by any living economy. This also means that while under a concave production sustainable development is ensured, this is no longer obvious under a convex-concave production. In reality, only industrialized economies can probably claim to have escaped their poverty traps, and therefore the possibility of observing a collapsing (non-sustainable) outcome, especially in less developed economies, does not have to be ruled out a priori.

[7] Ruling out the possibility of poverty traps means essentially to empty the basin of attraction of the trivial steady state, but this cannot be achieved for all the possible values of the parameters. It is surely possible to reduce the size of such a basin of attraction, and in this direction a reduction in the crucial parameter of the damage function, b, or in the degree of environmental inefficiency of productive activities, θ, can help. Note that affecting these parameters means intervening on fundamental aspects of the interaction between the economy and the environment.

[8] Note that all the qualitative conclusions discussed thus far in this section hold true even if we use the initial distribution of pollution $p(x,0)$ rather than the initial distribution of capital. However, if we stick to the reasonable hypothesis that $k(0,x)$ and $p(0,x)$ are proportional that is not a problem. Otherwise, if we think that $k(0,x)$ and $p(0,x)$ are not so simply related, the main conclusion would be similar, but we would need to consider several additional scenarios.

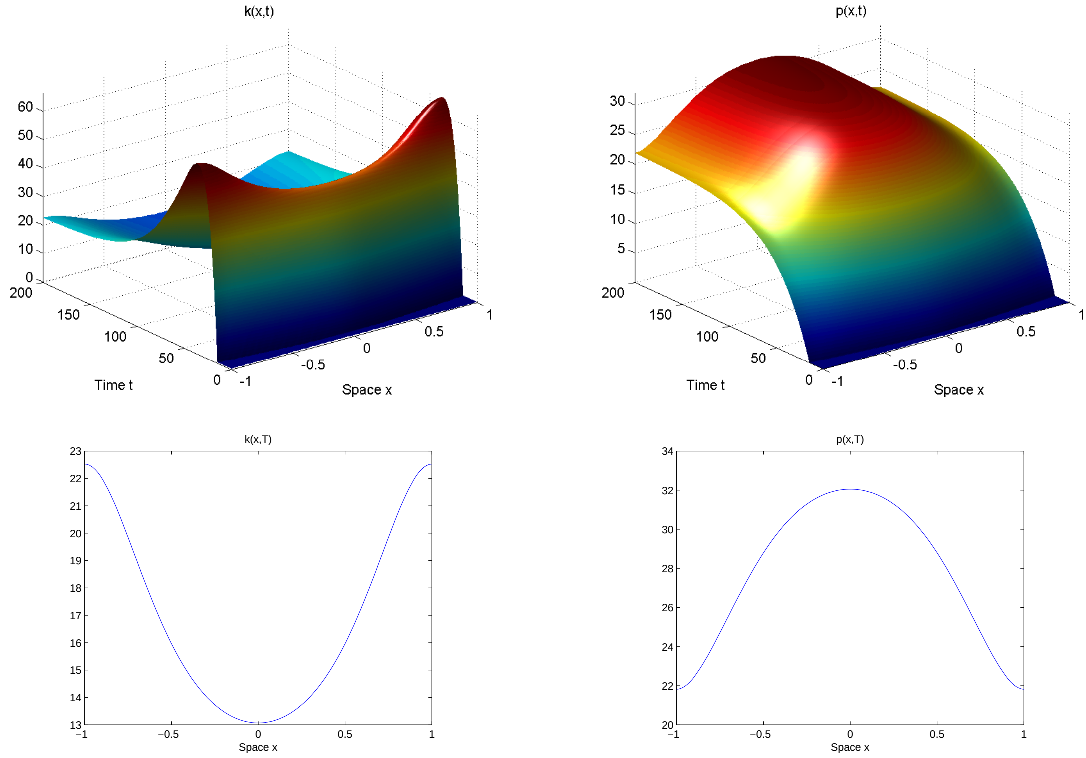

Figure 2.9: S-shaped production case: evolution of capital and pollution (with $k(x,0) = k(x,0)_l$ and $d = d_l$).

Finally, let us consider what happens whenever $u(x)$ is not homogeneous across space. We consider the case in which $d = d_l$ and $k(x,0) = k(x,0)_h$ and introduce a linear $u(x)$, as follows: $u(x) = 0.5x + 0.5$, meaning that u starts from zero and ends with one, linearly along the domain. We choose this scenario because diffusion is low enough for the structural differences across each location x not to be wiped out. As we can see in Figure 2.10 the regions close to the left boundary of the domain, that is regions where less attention has been placed on abatement activities, are those with greater long run values for capital, but these higher capital values are achieved at the cost of suffering higher pollution levels. By comparing Figure 2.7 with Figure 2.10 we can also see that, in the long run, the maxima and the minima of capital and pollution in the u linear case are respectively greater and lower than the corresponding values in the $u = 0.5$ case; this is due to the fact that we have exogenously introduced some heterogeneity in the abatement behavior.

2.4 The Ramsey-Type Problem

In this section we wish to relax the behaviorist hypothesis on economic and environmental policies: thus far $s(x)$ and $u(x)$ were just parameters (though space dependent ones). Now we let them to be optimally chosen by the social planner who by internalizing the spatial externalities selects the optimal spatio-temporal path of consumption $c(x,t)$ and share of abated emissions $u(x,t)$. Hence, as in the canonical Ramsey (1928) formulation, the social planner's

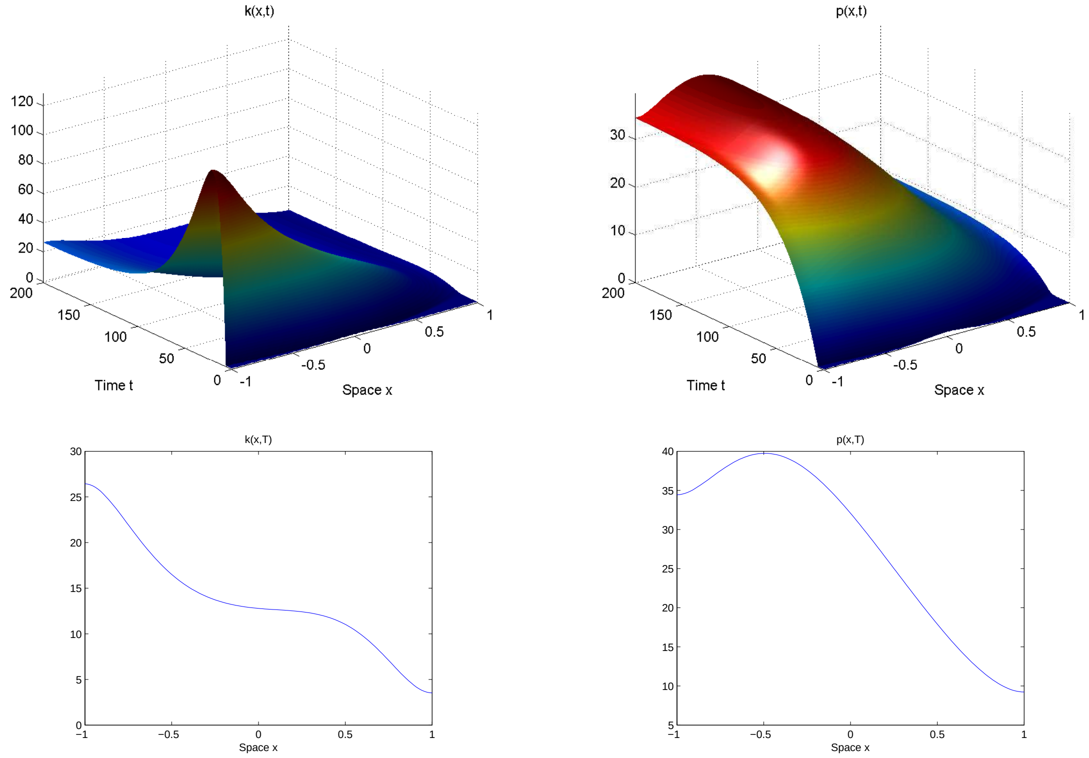

Figure 2.10: S-shaped production case: evolution of capital and pollution (with $k(x,0) = k(x,0)_h$ and $d = d_l$, along with $u(x) = 0.5 + 0.5x$).

problem can be expressed as follows:

$$\max_{c(x,t),u(x,t)} \quad \int_0^T \int_{x_a}^{x_b} [c(x,t) - \theta_p p(x,t)] e^{-\rho t} dx dt \tag{2.9}$$

$$s.t. \quad \frac{\partial k(x,t)}{\partial t} = d_1 \frac{\partial^2 k(x,t)}{\partial x^2} + \frac{f[k(x,t)][1-u(x,t)]^{\epsilon}}{a + bp(x,t)^2} - \delta_k k(x,t) - c(x,t). \tag{2.10}$$

$$\frac{\partial p(x,t)}{\partial t} = d_2 \frac{\partial^2 p(x,t)}{\partial x^2} + \theta \int_{x_a}^{x_b} [1 - u(x',t)] f[k(x',t)] \varphi(x',x) dx' - \delta_p p(x,t). \tag{2.11}$$

The utility function $U(x,t) = c(x,t) - \theta_p p(x,t)$ is separable and linear in consumption and pollution. This formulation can be seen as a generalization of the utility function adopted in Boucekkine et al. (2009), in which we have additionally introduced pollution as a disutility term. The parameter θ_p measures the relative importance of environmental quality with respect to consumption. Differently from Boucekkine et al. (2009), who consider an infinite time horizon, we do not focus on the very long run dynamics of the economic and environmental system, and we consider a finite horizon problem, in which the final horizon represents the amount of time needed to fully unfold environmental policy (e.g., suggested by international agreements). Moreover our model considers a finite domain in the spatial dimension, thus we do not need to introduce a spatial discount factor. It is worth to mention that an approach similar to ours in order to model growth and environmental issues has been suggested by Anita et al. (2013). We however extend their analysis by considering a second control variable, determining the extent of environmental protection activities.

We approach the optimal control problem by following the same variational method in Brock and Xepapadeas (2008, 2010). We obtain the generalized current value Hamiltonian function, $\mathcal{H}\big(k(x,t), p(x,t), u(x,t), c(x,t)\big)$, as follows:

$$\begin{aligned}\mathcal{H} &= c(x,t) - \theta_p p(x,t) + \lambda_k(x,t)\left[d_k \frac{\partial^2 k(x,t)}{\partial x^2} + \frac{f[k(x,t)][1-u(x,t)]^{\epsilon}}{a+bp(x,t)^2} - \delta_k k(x,t) - c(x,t)\right] + \\ &\quad +\lambda_p(x,t)\left[d_p \frac{\partial^2 p(x,t)}{\partial x^2} + \theta \int_{x_a}^{x_b} [1-u(x',t)] f[k(x',t)]\varphi(x',x)dx' - \delta_p p(x,t)\right].\end{aligned}$$

The optimality conditions are:

$$\begin{aligned}\frac{\partial \lambda_k(x,t)}{\partial t} &= \rho\lambda_k(x,t) - d_k \frac{\partial^2 \lambda_k(x,t)}{\partial x^2} - \lambda_k(x,t)\frac{f_k[k(x,t)][1-u(x,t)]^{\epsilon}}{a+bp(x,t)^2} + \delta_k \lambda_k(x,t) + \\ &\quad -[1-u(x,t)]f_k[k(x,t)]\theta \int_{x_a}^{x_b} \lambda_p(x',t)\varphi(x',x)dx'. \end{aligned} \tag{2.12}$$

$$\frac{\partial \lambda_p(x,t)}{\partial t} = \rho\lambda_p(x,t) - d_p \frac{\partial^2 \lambda_p(x,t)}{\partial x^2} + \theta_p + \lambda_k(x,t)\frac{2bp(x,t)f[k(x,t)][1-u(x,t)]^{\epsilon}}{[a+bp(x,t)^2]^2} + \delta_p \lambda_p(x,t), \tag{2.13}$$

while for the control variables, the maximum principle reads as:

$$\underset{c(x,t),u(x,t)}{Max} \mathcal{H}\big(k(x,t), p(x,t), u(x,t), c(x,t)\big). \tag{2.14}$$

We need to solve a system of four backward-forward partial differential equations, namely equations (2.10), (2.11), (2.12), (2.13), where the controls variables $u(x,t)$ and $c(x,t)$ are selected via the maximum principle (2.14), and the initial conditions on the state variables $k(x,t)$ and $p(x,t)$, the final conditions on the costate variables $\lambda_k(x,t)$ and $\lambda_p(x,t)$ and the Neumann conditions on the first derivatives are given below:

$$\begin{aligned}&k(0,x) = k_0(x); p(0,x) = p_0(x).\\ &k(0,x) = k_0(x) = k_0 e^{-\frac{x^2}{k_0}}.\\ &p(0,x) = p_0(x) = p_0 e^{x}.\\ &\lambda_k(x,T) = 0.\\ &\lambda_p(x,T) = 0.\\ &\frac{\partial k(x_a,t)}{\partial x} = \frac{\partial k(x_b,t)}{\partial x} = \frac{\partial p(x_a,t)}{\partial x} = \frac{\partial p(x_b,t)}{\partial x} = 0 \; \forall t \in [0,T].\\ &\frac{\partial \lambda_k(x_a,t)}{\partial x} = \frac{\partial \lambda_k(x_b,t)}{\partial x} = \frac{\partial \lambda_p(x_a,t)}{\partial x} = \frac{\partial \lambda_p(x_b,t)}{\partial x} = 0 \; \forall t \in [0,T].\end{aligned}$$

As in the previous section, we consider two different production functions, the S-shaped and the Cobb-Douglas ones, with the latter as a benchmark case. With respect to previous section, the only additional conditions we need to impose involve the control variables, and they read as follows:

$$0 \leq c(x,t) \leq f[k(x,t)]\frac{[1-u(x,t)]^{\epsilon}}{a+bp(x,t)^2}. \tag{2.15}$$

$$0 \leq u(x,t) \leq 1. \tag{2.16}$$

The condition on consumption in (2.15) states that gross investment should not be negative at any time and at any location; (2.16) states instead the obvious condition that the abatement share needs to belong to the closed interval $[0,1]$.

2.4.1 Spatio-Temporal Dynamics

For the numerical solution of our optimal control problem, we rely on a generalization of the Sweep Algorithm (see McAsey et al., 2012; and see also appendix 2.6 for further technical details). Finally, the parameters and functions employed in the simulations are the following:

$$\begin{cases} \theta = 0.2, \epsilon = 1, \alpha_1 = 1, \alpha_2 = 1, \theta_p = 1, x_a = -1, x_b = 1, \\ \delta_k = 0.001,\ \delta_p = 0.05,\ a = 1,\ b = 0.01, \\ A = 100,\ q = 4,\ d_k = d_p = 1,\ k_0 = 1,\ p_0 = 1, \\ k(x,0) = k_0 e^{-\frac{x^2}{k_0}},\ p(x,0) = p_0 e^{x},\ \text{and } \varphi(x'x) = \delta(x',x). \end{cases} \tag{2.17}$$

The choice of the above parameters follows the same clarity purpose as we adopted in the previous section. There are minor differences with respect to the initial capital and pollution distribution employed in the previous section, but they do not change our qualitative results.

In the Cobb-Douglas case, as illustrated in Figure 2.11, the optimal consumption c is zero in any position at the beginning of the planning horizon, while taxation u is at its maximum value, $u = 1$. Such an initial outcome persists until, abruptly, the optimal consumption reaches its maximum value and the optimal taxation drops to zero. In other words, it is optimal to keep consumption low and taxation high initially, at any location in the spatial economy, and then do exactly the opposite, as the end of the planning horizon approaches. The optimal controls affect the evolution of the state variables: the high level of initial abatement activities drains resources from the accumulation of capital, which have limited dynamics but ends with a positive steady state for all the regions on the domain. Given the spatial homogeneity of the controls, pollution remains higher in the more initially polluted regions. It is to be noticed that when the optimal policy switches from $c = 0$ and $u = 1$ to $c = 1$ and $u = 0$ there is an increase in the level of pollution in each region and the opposite happens to the level of capital: of course this is due to the simulataneous change in the optimal policies.

As is it possible to observe in Figure 2.12, for the S-shaped case, we have a qualitatively similar behavior of the optimal consumption level and taxation rate, but we gain in the spatial heterogeneity: the profiles of the optimal policies in space and time show that the regions starting with a higher distribution of pollution ($p(0,x) = p_0 e^x, p_0 = 1$) are the last experiencing a reduction in the abatement taxation, from the maximal level, $u = 1$, to the minimal

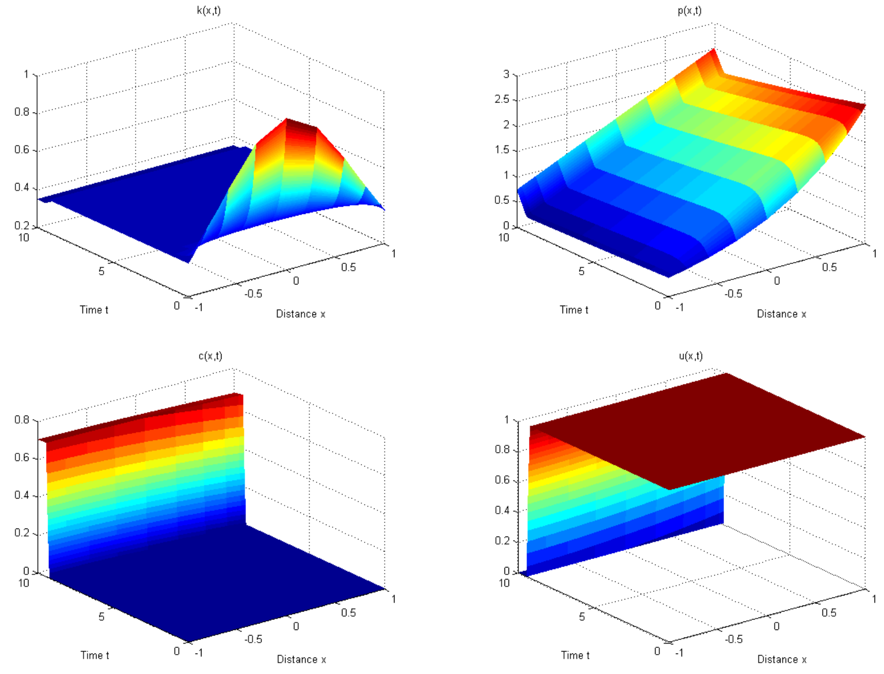

Figure 2.11: Cobb-Douglas production case: evolution of capital, pollution, consumption and share of abated emissions in the optimally planned case (with $d = 1$ and $k_0 = 1$).

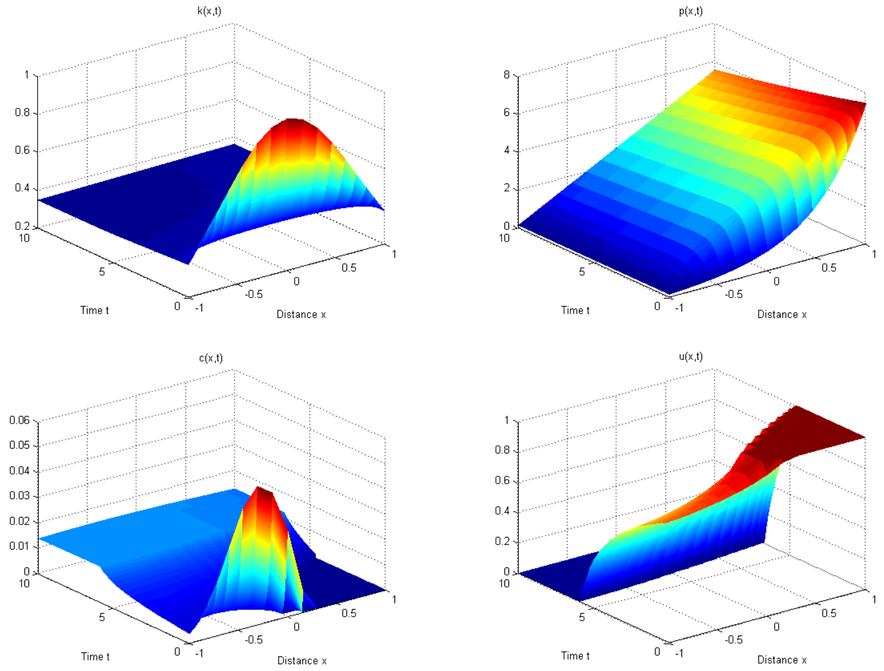

Figure 2.12: S-shaped production case: evolution of capital, pollution, consumption and share of abated emissions in the optimally planned case (with $d = 1$ and $k_0 = 1$).

level, $u = 0$, and, accordingly, the last to appreciate an increase in the optimal consumption policy c, from $c = 0$ to $c = c(T, x)$. On the contrary, the regions on the left (i.e., the initially least polluted regions), have a starting level of abatement taxation $u < 1$, and suffer taxation for a shorter period of time, while the optimal consumption is strictly positive and increasing since the beginning of the planning horizon. As in the previous case, the optimal policies shape the spatio-temporal dynamics of capital and pollution: capital has again limited temporal dynamics but heterogeneous (and positive) spatial profile at the end of the period T, where $k(T, x)$ increases with the level of pollution $p(T, x)$ from the left to the right. As before, the level of pollution remains higher in the more initially polluted regions, but in these regions there is actually a decrease in p, while the least initially polluted regions pay the price of consumption with a higher final value of p. It is important to stress that the spatial economy reaches a positive steady state, meaning that given the parameter values and the initial distribution of capital and pollution, in the optimal control framework policymakers are able to address the economy toward a sustainable outcome. That is not obvious at all in the S-Shaped production function case, as already discussed in the previous section.

2.5 Conclusions

In order to shed some light on how the evolution of capital and pollution interact not only over time but also across space, this paper studies an economic growth model with pollution abatement activities. Specifically we extend the spatial model by Camacho and Zou (2004) and Boucekkine et al. (2009) in order to allow for pollution diffusion (Camacho and Pérez–Barahona, 2015). Our framework introduces a kernel-based approach to modeling the spatial spread of pollution, allowing to distinguish between diffusive and non-diffusive spatial externalities, contrasting the outcomes under a neoclassical and a convex-concave production function. Differently from what happens with a globally concave production function, the case of an S-shaped production technology since allowing for eventual poverty traps exhibits a richer variety of possible scenarios, mainly due to the fact that its evolution operator is non-concave. In addition, the behavior of capital and pollution are dependent upon the initial condition and the diffusivity parameters. In particular, our simulations show that in some cases in the long run the economy survives in the presence of a nonzero bounded level of pollution, but it could even be the case that the presence of a negative pollution feedback on output leads every region in the spatial economy to collapse. We show that in the convex-concave production framework the spatial implications of capital and pollution, thanks to two different channels (namely, diffusion and pollution externality), might allow poor regions to escape their poverty traps (or alternatively, it might condemn also rich regions to collapse in the long run). Specifically, whenever rich regions are substantially rich diffusion can play a very important role in allowing poor regions to escape their poverty traps; if however they are not rich enough diffusion might condemn also rich regions to collapse in the long run. However, even if rich regions are particularly rich but pollution feeds back on economic production strongly enough, the whole spatial economy might be doomed to collapse.

Note that our paper represents only a preliminary attempt to analyze the mutual relationship between capital accumulation and pollution in a spatial framework. However, the problem of sustainable development is much more complex than what we could describe in our model thus further research is definitely needed in order to shed some more light on the potential spatial heterogeneities generated by environmental policies and issues. For example, in our analysis we have not considered the effect of cleaning technological progress, thus it is natural wondering how our results might differ by introducing such s further complication in the joint dynamics of capital and pollution. In particular, the arising of an environmental Kuznets curve (as shown in Brock and Taylor, 2010) which is generally interpreted as a positive outcome for the economic and environmental system could be totally prevented by the diffusion of pollution from highly to lowly polluted locations. Thus, in a spatial context the implications of traditional (a-spatial) environmental policies might need to be substantially revised. This challenging task is left for future research.

2.6 Appendix: Numerical Simulations

We implement the forward-backward sweep method for the system (2.10), (2.11), (2.12), (2.13) and (2.14) as follows:

1. We start by choosing an initial guess $(c^{(0)}, u^{(0)}) = (c^{(0)}(t), u^{(0)}(t))$.

2. Iteration for $j \geq 0$: by using the spectral method, we solve

$$\frac{\partial k^{(j+1)}(x,t)}{\partial t} = d_k \frac{\partial^2 k^{(j+1)}(x,t)}{\partial x^2} + \frac{g[k^{(j+1)}(x,t)][1-u^{(j)}(x,t)]^{\epsilon}}{a+bp^{(j+1)}(x,t)} - \delta_k k^{(j+1)}(x,t) - c^{(j)}(x,t),$$

$$\frac{\partial p^{(j+1)}(x,t)}{\partial t} = d_p \frac{\partial^2 p^{(j+1)}(x,t)}{\partial x^2} + \theta \int_{x_a}^{x_b} [1-u^{(j)}(x',t)] f[k^{(j+1)}(x',t)] \varphi(x',x) dx' - \delta_p p^{(j+1)}(x,t),$$

subject to

$$\begin{aligned} k^{(j+1)}(x,0) &= k_0(x) \text{ in } \Omega, \\ p^{(j+1)}(x,0) &= p_0(x) \text{ in } \Omega, \\ \frac{\partial k^{(j+1)}(x,t)}{\partial n} &= 0 \text{ on } \partial\Omega, \\ \frac{\partial p^{(j+1)}(x,t)}{\partial n} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

We reverse equation (2.12) and (2.13) in time, via the change of variable $\bar{t} = T - t$, turning the problem into a forward problem with zero initial conditions. Then, we solve

$$\begin{aligned} \frac{\partial \lambda_k^{(j+1)}(x,t)}{\partial t} &= \rho\lambda_k^{(j+1)} - d_k \frac{\partial^2 \lambda_k^{(j+1)}(x,t)}{\partial x^2} - \lambda_k^{(j+1)}(x,t) \frac{f_k[k^{(j+1)}(x,t)][1-u^{(j)}(x,t)]^{\epsilon}}{a+bp^{(j+1)}(x,t)^2} + \\ &\quad + \delta_k \lambda_k^{(j+1)}(x,t) - [1-u^{(j)}(x,t)] f_k[k^{(j+1)}(x,t)] \theta \int_{x_a}^{x_b} \lambda_p^{(j+1)}(x',t) \varphi(x',x) dx' \\ \frac{\partial \lambda_p^{(j+1)}(x,t)}{\partial t} &= \rho\lambda_p^{(j+1)} - d_p \frac{\partial^2 \lambda_p^{(j+1)}(x,t)}{\partial x^2} + \theta_p + \lambda_k^{(j+1)}(x,t) \frac{2bp^{(j+1)}(x,t) f[k^{(j+1)}(x,t)][1-u^{(j)}(x,t)]^{\epsilon}}{[a+bp^{(j+1)}(x,t)^2]^2} + \\ &\quad + \delta_p \lambda_p^{(j+1)}(x,t) \end{aligned}$$

subject to

$$\lambda_k^{(j+1)}(x,0) = 0 \text{ in } \Omega,$$
$$\lambda_k^{(j+1)}(x,0) = 0 \text{ in } \Omega,$$
$$\frac{\partial\lambda_k^{(j+1)}(x,\bar{t})}{\partial n} = 0 \text{ on } \partial\Omega,$$
$$\frac{\partial\lambda_k^{(j+1)}(x,\bar{t})}{\partial n} = 0 \text{ on } \partial\Omega.$$

Using MATLAB fmincon function (function is used to find the minimum of constrained nonlinear multivariable function) defined below

$$\min_x f(x) \text{ such that } \begin{cases} c(x) \leq 0 \\ ceq(x) = 0 \\ A \cdot x \leq b \\ Aeq \cdot x = beq \\ lb \leq x \leq ub, \end{cases}$$

to find the value of $c(x,t)$ and $u(x,t)$ that maximize $\mathcal{H}$. We achieve this, by finding the value of $c(x,t)$ and $u(x,t)$ that minimize $-\mathcal{H}$. Finally we check for convergence by computing the difference between the values of $c(x,t)$ and $u(x,t)$ in this iteration and the corresponding ones in the last iteration. If the $\mathcal{L}^2$-norm of the difference is negligibly small, we output the current function as solution, otherwise we continue iterating.

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Chapter 3

Sustainability and Intertemporal Equity: A Multicriteria Approach

3.1 Introduction

Traditionally macroeconomics wishes to assess the impact of different events or policies on the wellbeing of the society as a whole. This is critical since it requires to understand how to define social welfare. The approach widely used in literature relies on the so-called discounted utilitarianism. According to such a criterion, social welfare coincides with the discounted sum of instantaneous per capita utilities. Such an approach is very convenient since it allows to state a canonical macroeconomic problem as an optimal control problem in which the objective function, coinciding with social welfare, is bounded and thus standard mathematical techniques can be borrowed and directly applied to find its solution. However the recent growing interest towards sustainability raises several questions about the effective ability of discounted utilitarianism to provide meaningful investigations of the underlying problems[1] (see Heal, 2005). Even if the notion of sustainability is still controversial, the most widely spread definition has been provided by the World Commission on Environment and Development which labels sustainable development as that kind of development satisfying *"the needs of the present without compromising the ability of future generations to meet their own needs"* (WCED, 1987). As it may be clear from such a definition, any sustainability discourse requires to carefully take into consideration two different aspects: respect of natural resources and intertemporal equity (Chichilnisky et al., 1995). Thus, the discounting utilitarianism, requiring to discount at a certain (generally constant) rate instantaneous utilities, is not compatible with intergenerational equity since it attaches less weight to future generations. Almost one century ago Ramsey (1928) recognized that *"discounting of future utilities is ethically indefensible and arises purely from a weakness of the imagination"* (Ramsey, 1928).

[1] Another critical aspect associated to the use of the utilitarian approach is related to the role of the population size and its eventual growth. Specifically, two different utilitarian approaches have been proposed in literature, the average (welfare coincides with individual or average utility) and total (welfare is the sum of individual utilities across the population) utilitarianism. See Palivos and Yip (1993) or more recently Marsiglio and La Torre (2012), Boucekkine and Fabbri (2013), and Marsiglio (2014) for a discussion of the implications of different utilitarian approaches. Since we abstract from population growth and normalize the population size, in our paper average and total utilitarianism coincide, thus we do not explicitly relate to this branch of the literature.

In order to fix the shortcoming related to discounted utilitarianism, several proposals have been advanced[2] (Ramsey, 1928; von Weizcker, 1967). Probably the most interesting and discussed approaches are the green golden rule and the Chichilnisky criterion. The green golden rule defines social welfare as the asymptotic utility level thus allowing to determine the highest indefinitely maintainable utility level (Chichilnisky et al., 1995). The Chichilnisky criterion defines social welfare as a weighted average between the discounted utilitarianism and the green golden rule welfare (Chichilnisky, 1997). Despite its simplicity this latter notion generates several problems in order to identify optimal paths[3] (Figuieres and Tidball, 2012), thus understanding how to assess sustainability outcomes is still an open question. The goal of this paper is to shed some light on this issue by developing a multicriteria approach which will allow us to compare how the definition of social welfare might impact on the outcomes of alternative economic choices. We thus consider a simple model with natural resources and environmental concern, as in Chichilnisky et al. (1995), to assess the performance of these criteria to achieve sustainable outcomes. Differently from Chichilnisky et al. (1995), we abstract from capital accumulation and focus only on the interactions between consumption choices and the evolution of natural assets in order to emphasize the trade-off between the (short run) economic benefits and the (long run) environmental costs associated to human activities.

This paper proceeds as follows. Section 3.2 proposes our macroeconomic model, which is simply an optimal control problem with one state and one control variable, plus a scrap function depending on both the control and state variable (consumption level and stock of natural resources, respectively). Section 3.3 briefly reviews the rationale behind multicriteria analysis and focuses on the two approaches, namely the scalarization and the goal programming techniques, which we will then use to translate our dynamic macroeconomic problem into a multicriteria model. This is explicitly done in section 3.4, where we develop both the scalarized and goal programming versions of the multicriteria model we employ in order to assess the impact of alternative notions of social welfare on consumption, natural resources and welfare level; we also discuss the results of our analysis for a certain parametrization of the model, and compare the outcomes under the two alternative specifications of the multicriteria problem. We show that for a realistic set of parameter values a clear ranking (in terms of welfare achievements) of welfare criteria exists: the green golden rule yields the highest level of welfare, the Chichilnisky criterion an intermediate level, and the discounted utilitarianism the lowest welfare level. Section 3.5 as usual concludes and proposes directions for future research.

3.2 The Model

According to the traditional macroeconomic literature, we consider a Ramsey-type (1928) model of optimal growth where a benevolent social planner tries to maximize social welfare. Social welfare, W, is defined according to the Chichilnisky criterion (Chichilnisky, 1997), meaning that it is a weighted average between the discounted utilitarian and green golden rule approach. Time is continuous, the time horizon is infinite and the pure rate of time preference is denoted by $\rho > 0$; the instantaneous utility function depends on the level of consumption, c_t, and the stock of natural

[2]Several other criteria have been proposed in literature but to a large extent they turn out to be ad hoc proposal or do not allow a direct comparison with discounted utilitarianism (Pezzey, 1997; Arrow et at., 2004; Marsiglio, 2011).

[3]Another interesting related work is Le Kama's (2001), showing that by choosing the green golden rule utility level as Ramsey's bliss point for the non-discounted problem the optimal utilitarian path converges to the green golden rule outcome.

resources, e_t, and it is assumed to take the following isoelastic form: $u(c_t, e_t) = \frac{(c_t e_t^{\beta})^{1-\sigma}-1}{1-\sigma}$, where $\sigma > 0$ is the inverse of the intertemporal elasticity of substitution and $\beta \geq 0$ represents the weight of the environment in the planner's preferences (the green preferences parameter). Population is constant and its size is normalized to 1, thus aggregate and per capita variables coincide. Natural resources accumulate according to their renewal capacity, assumed to be logistic, but are depleted by consumption activities; the rate of natural regeneration is r and e^c denotes the carrying capacity of the environment (Chichilnisky et al., 1995). Given the initial level of natural resources, e_0, the planner's problem consists of choosing the consumption level in order to maximize social welfare by taking into account the dynamic evolution of natural assets:

$$\max_{c_t} \qquad W = \theta \int_0^{\infty} \frac{c_t^{1-\sigma} e_t^{\beta(1-\sigma)} - 1}{1-\sigma} e^{-\rho t} dt + (1-\theta) \lim_{t \to \infty} \frac{c_t^{1-\sigma} e_t^{\beta(1-\sigma)} - 1}{1-\sigma} \tag{3.1}$$

$$s.t. \qquad \dot{e}_t = r e_t \left(1 - \frac{e_t}{e^c}\right) - c_t \tag{3.2}$$

Note that the objective function (3.1) corresponds to the Chichilnisky criterion, in which social welfare is defined as the weighted average between the discounted utilitarian welfare ($\int_0^{\infty} u(c_t, e_t) e^{-\rho t} dt$) and the green golden rule welfare ($\lim_{t\to\infty} u(c_t, e_t)$). The parameter $\theta \in [0, 1]$ represents the weight assigned to the discounted integral of utilities. Specifically, when $\theta = 1$ ($\theta = 0$) social welfare is defined according to the discounted utilitarian (green golden rule) criterion, while for any $\theta \in (0, 1)$ social welfare takes into account both the discounted utilitarian and green golden rule approach. Since analytical solutions cannot be found (unless in the extreme cases in which $\theta = 0$ or $\theta = 1$), in order to understand the role of the parameter θ in determining the evolution path of consumption and natural resources, and thus the social welfare level, we need to rely on a multicriteria approach.

As we will see later, the problem above turns out to be a particular case of a more general problem that can be tackled with the help of the multiple criteria decision analysis, in which two alternative (the discounted utilitarian and green golden rule) criteria are simultaneously pursued. Specifically, the maximization problem in (3.1) and (3.2) can be recast as the following multicriteria problem:

$$\max \qquad W = [W_{DU}, W_{GGR}], \tag{3.3}$$

where W_{DU} and W_{GGR} represent the discounted utilitarian (DU) and green golden rule (GGR) welfare criteria respectively, defined as follows:

$$W_{DU} \equiv \int_0^{\infty} \frac{c_t^{1-\sigma} e_t^{\beta(1-\sigma)} - 1}{1-\sigma} e^{-\rho t} dt \tag{3.4}$$

$$W_{GGR} \equiv \lim_{t \to \infty} \frac{c_t^{1-\sigma} e_t^{\beta(1-\sigma)} - 1}{1-\sigma}, \tag{3.5}$$

and the maximization problem is subject to the dynamical constraint in (3.2). This ability to transform our macroeconomic model into a simple multicriteria problem is very convenient since it allows us to borrow from the operational research literature in order to understand the impact of alternative notions of social welfare (i.e., different values of θ)

on consumption, natural resources and welfare levels.

3.3 Multicriteria Decision Analysis

Multiple criteria decision analysis (MCDA, also known as multiple criteria decision making, MCDM) explicitly considers multiple and conflicting criteria such as cost, price, quality, time, performance, and others, in complex decision-making contexts and provides an alternative to the classical cost-benefit analysis (a popular tool in the 1970s and the 1980s), extensively used in economics in order to compare alternative policies or projects. Decision aid tools aim at establishing formulations of propositions to be submitted to the judgment of a decision maker or a group of decision makers (the social planner in our macroeconomic framework). As in any decision, since we have to consider different points of view and perspectives (dealing with finance, human resources, security, quality, etc.), a multicriteria approach appears more suitable to describe all different components of a decision-making process. Indeed, structuring complex problems as multiple criteria models leads to take better decisions. Since the beginning of the modern MCDA discipline in the early 1960s, many approaches and methods have been developed, also supported by many advanced and computationally-efficient decision-making software. MCDA methods are applied wherever there are several alternatives, which must be ranked in accordance with their significance in respect with the aim of the research or where the best alternative among the available ones must be identified. In other words, the primary purpose of analysis is a search for a compromise solution (Guitoni and Martel, 1998). By focusing only on projects that involve compromises between environmental costs and economic benefits, applications of MCDA in different areas have been proposed, such as industrial development (Nijkamp and van Delft, 1977), environmental policy issues (Janssen, 2001; Gamper and Turcanu, 2007), sustainability assessment at macroeconomic level (Shmelev, 2011), and macroeconomic policy (André et al., 2009).

There exist several alternatives to deal with a MCDA context (see for example Sawaragi et al., 1985; and Steuer, 1986); in the sequel we will mainly focus on two of them, namely the scalarization technique and the goal programming (GP) approach. The former method, namely the scalarization technique, represents the easiest way to deal with a MCDA model. Scalarizing a MCDA problem consists of constructing a single-objective optimization problem such that optimal solutions of the single-objective optimization problem are the Pareto optimal solutions of the MCDA problem (Hwang and Masud, 1979). Scalarizing functions play an essential role in this decision-making context: in literature several different scalarizing functions have been proposed based on different approaches and philosophies. Probably the simplest specification is the weighted scalarization, which simply attaches a specific weight to each of the criteria. Mathematically, if $f_1(x), f_2(x), ..., f_p(x)$ are p criteria to be maximized, and $w_i \in [0, 1]$ are weights such that $\sum_{i=1}^{p} w_i = 1$, a weighted scalarized model takes the form:

$$\max \qquad \sum_{i=1}^{p} w_i f_i(x)$$

The optimal solution of the MCDA problem will be depending on the chosen weights, and vary with them. The latter approach, that is the GP technique, is one of the most well-known MCDA models in which the solution of the best compromise minimizes the absolute deviations between the achievement levels $f_i(x)$ and the aspiration levels g_i, $\forall i = 1, ..., p$. In fact, given a certain aspiration level both positive, δ_i^+, and negative, δ_i^+, deviations are unwanted. It is an a priori method, which means that the information regarding the goals is first decided by the decision maker and then a solution is determined by minimizing the difference between the achievement levels and the corresponding goals. In other words the original objectives of the problem are transformed into constraints and the optimization of the deviations from the goals results indirectly to optimize the initial objectives. The first formulation of the GP model has been presented by Charnes et al. (1955), and Charnes and Cooper (1959, 1968). Among all different GP formulations the weighted GP model with satisfaction function is the most suitable for our purposes. In this framework the decision maker (social planner) compares the performances of every possible action through the satisfaction function $F_i(\delta_i)$ (for more details see Martel and Aouni, 1990). The decision maker wishes to maximize his/her satisfaction, thus the greatest deviations are associated with lowest degrees of satisfaction. The mathematical formulation of the weighted GP program with satisfaction function can be expressed as follows:

$$\begin{aligned} \max \qquad & Z = \sum_{i=1}^{p} \left[w_i F_i^+(\delta_i^+) + w_i F_i^-(\delta_i^-) \right] \\ s.t. \qquad & f_i(x) - \delta_1^+ + \delta_1^- = g_i \\ & 0 \leq \delta_i^+ \leq \delta_{i\nu}^+, \quad \forall i = 1, ..., p \\ & 0 \leq \delta_i^- \leq \delta_{i\nu}^-, \quad \forall i = 1, ..., p \end{aligned}$$

In the above specification the parameters $\delta_{i\nu}^+$ and $\delta_{i\nu}^-$ represent veto thresholds, such that both positive and negative deviations, δ_i^+ and δ_i^-, cannot exceed such threshold values, $\delta_{i\nu}^+$ and $\delta_{i\nu}^-$ respectively.

Both the scalarization and GP approaches can be used in order to define a multicriteria problem, allowing to deal with a model like ours in (3.1) and (3.2). They both are easy to implement and can be solved through some powerful mathematical programming software such as LINGO and CPLEX. However, they both have pros and cons. Specifically, a positive aspect of goal programming is its simplicity and ease of use, specially because of its applicability to real problems (Aouni and Kettani, 2001). Moreover, when the parameters are subject to noise or uncertainty the problem can be easily analyzed by relying on stochastic GP models (see Aouni et al. 2012a, 2012b, 2013; Aouni and La Torre, 2010). With respect to the weighted scalarization approach, goal programming is not always able to produce solutions that are Pareto optimal. In fact, the GP model is based on the so-called “satisfying philosophy”, meaning that the optimization task is replaced by the satisfaction in reaching certain levels for each criterion.

We will use both these approaches to analyze our macroeconomic problem, showing that they will lead to qualitatively equivalent results. The multicriteria optimization problem, both it its scalarized and GP versions, is solved with LINGO 14; given the dynamic nature of the problem and thus its large number of variables, we do not report LINGO’s solution in our paper but we summarize it by showing the (optimal) dynamic evolution of our control (consumption) and state (natural resources) variables.

3.4 A Multicriteria Approach

We now propose two different methods in order to deal with our problem in (3.1) and (3.2), based on the weighted scalarization technique and the weighted GP with satisfaction function, respectively. We first show how to use these two alternative methodologies in order to study our problem and then compare the solution obtained under these two formulations of the multicriteria problem. Note that, since in the dynamical problem (3.3) time is continuous, in order to perform our numerical simulations we need firstly to discretize the problem. Moreover, since the time horizon is infinite, we also approximate this by assuming that time is finite, with the final time T being sufficiently remote in the future to be actually seen as a plausible simplification of an infinite horizon. Since the utility contribution to the integral decays exponentially with time, a final time as $T = 100$ is enough to appropriately do so. It is however possible to show that the results are robust with respect to the choice of the final time T. Since our main goal in the paper consists of assessing the impact of the different notions of social welfare on the dynamics of consumption, natural resources and welfare level, in both the formulations of the problem, we emphasize the role of the parameter θ which allows us to distinguish the discounted utilitarian ($\theta = 1$), the green golden rule ($\theta = 0$) and the Chichilnisky ($0 < \theta < 1$) criteria. Specifically, we analyze the behavior of the solution for different values of θ in the range $\theta \in [0, 1]$.

A critical aspect of any simulation is related to the value of the parameters employed. When possible we rely on standard values traditionally used in macroeconomics literature, while when not possible we rely on broad estimates of the parameters. Specifically, the value of the parameters used in our simulations is summarized in Table 3.1.

σ	ρ	β	r	e^c	e_0
2	0.04	0.2	0.05	3	1

Table 3.1: Parameter values employed in our simulations.

The pure rate of time preference, ρ, and the inverse of the intertemporal elasticity of substitution, σ, are set equal to 0.04 and 2, respectively (Barro and Sala-i-Martin, 2004). The green preference parameter, β, may take values between 0.1 and 0.3, according to different degrees of environmental concern (see Bartz and Kelly, 2008). We thus choose an average between these two values, that is $\beta = 0.2$. Note however that different values of the green preference parameter would yield the same qualitative results, apart from the extreme case $\beta = 0$ (no environmental concern), which is not interesting for our analysis. The parameter values corresponding to the stock of natural resources strongly depend on the specific resource we are analyzing. Since we focus on a wide concept of natural resources, thus encompassing a broad range of natural assets, we rely on broad estimates related to fishery and forestry which can describe quite well the notion of natural resource we are adopting in our model. The initial value of the stock of natural resources, e_0, is set equal to 1 for the sake of simplicity. The carrying capacity, e^c, is set equal to 3; such a value is chosen since being relatively close to the initial stock allows to better stress the nature of the economic and environmental trade off. Nevertheless the results are robust for different values of the carrying capacity parameter. We assume the rate of natural regeneration, r, to be equal to 0.05 (see Eliasson and Turnovsky, 2004).

3.4.1 Scalarization Technique

According to the scalarization technique, we scalarize the vector $W = [W_1, W_2]$ with the help of two weights, such that what we need to optimize is a traditional single-criterion problem:

$$\max_{c_t} \qquad W = w_{DU}W_{DU} + w_{GGR}W_{GGR}$$

where $w_{DU} = \theta$ and $w_{GGR} = 1 - \theta$, represent the relative importance the decision maker (the social planner) assigns to each objective W_i, $i = DU, GGR$, while W_{DU} and W_{GGR} are defined in (3.4) and (3.5), respectively. Note that this specification is exactly equivalent to our problem in (3.1) and (3.2), in which the objective function corresponds to the Chichilnisky criterion. By proceeding with the discretization, the infinite integral in (3.4) is substituted with a finite sum, while the limit in (3.5) with the value taken by the utility function at the final time T; the dynamic constraint instead is substituted with a set of difference equations. Hence the problem to be solved numerically is the following:

$$\max_{c_t} \qquad W = \theta \sum_{t=0}^{T} \frac{c_t^{1-\sigma} e_t^{\beta(1-\sigma)} - 1}{1-\sigma} e^{-\rho t} + (1-\theta) \frac{c_T^{1-\sigma} e_T^{\beta(1-\sigma)} - 1}{1-\sigma} \tag{3.6}$$

$$s.t. \qquad e_{t+1} - e_t = r e_t \left(1 - \frac{e_t}{e^c}\right) - c_t \quad \forall t = 0, ...T-1 \tag{3.7}$$

$$0 = r e_T \left(1 - \frac{e_T}{e^c}\right) - c_T \tag{3.8}$$

$$e_0 \text{ given} \tag{3.9}$$

Some comments on the equations (3.6) to (3.9) are needed. When the final time T is reached the economy is frozen, meaning that a steady state has been achieved and thus the second addendum in the equation (3.6) can be interpreted as lasting forever, approximating the idea of the maximum indefinitely substainable level of utility. Therefore there is no dynamics after T, as described by the equation (3.8).

The results of our numerical simulations are shown in Figure 3.1 and Figure 3.2. Figure 3.1 shows the optimal path of consumption and natural resources for different values of θ, ranging from 0 to 1, corresponding to social welfare functions relying on different approaches, varying from the green golden rule ($\theta = 0$) to the discounted utilitarianism ($\theta = 1$). The intermediate values of θ correspond to the Chichilnisky criterion, in which higher values of θ represent more importance attached to discounted utilitarianism. As the figure clearly shows, the closer θ to 0, the more parsimonious is the consumption path and the more abundant the long run level of natural resources. Intuitively, the more emphasis we place on the steady state utility, the more we favor environmental preservation rather than economic activities; we thus consume less and preserve more natural resources for the (remote) future. Note that in the figure, the green golden rule case ($\theta = 0$) is reported with a dashed line in order to remind us that this represents only a steady state level, thus there is no dynamics from 0 to T.

Figure 3.2 represents the value of social welfare, W (equation (3.6)), as a function of the parameter $\theta \in [0, 1]$. Social welfare is clearly decreasing in θ, meaning that it achieves its maximum when $\theta = 0$, corresponding to the green golden rule case, and its minimum when $\theta = 1$, corresponding to the discounted utilitarian case. Thus the green golden

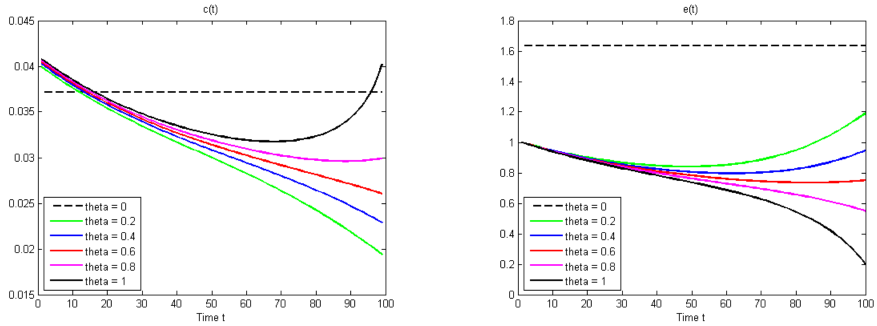

Figure 3.1: Scalarization technique: evolution of consumption (on the left) and natural resources (on the right) for different values of θ.

rule approach does yield a higher social welfare than both the Chichilnisky criterion and the discounted utilitarian approach. This result suggests that completely neglecting finite-time utilities and focusing only on the steady state utility, as suggested by the green golden rule, is not only sensible from a sustainability point of view but it is also from a social welfare maximization standpoint.

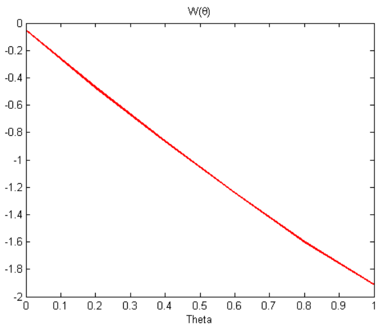

Figure 3.2: Scalarization technique: social welfare as a function of θ.

3.4.2 Goal Programming Technique

According to the chosen GP technique, the preferences on the deviations from the fixed aspiration levels are explicitly taken into account with the help of satisfaction functions $F_i(\delta_i)$, and our multicriteria problem can be expressed as the following maximization problem:

$$\begin{aligned}
\max \quad & w_{DU}F^+_{DU}(\delta^+_{DU}) + w_{DU}F^-_{DU}(\delta^-_{DU}) + w_{GGR}F^+_{GGR}(\delta^+_{GGR}) + w_{GGR}F^-_{GGR}(\delta^-_{GGR}) \\
s.t. \quad & W_{DU} - \delta^+_{DU} + \delta^-_{DU} = G_{DU} \\
& W_{GGR} - \delta^+_{GGR} + \delta^-_{GGR} = G_{GGR} \\
& 0 \leq \delta^+_{DU} \leq \delta^+_{DU\nu} \\
& 0 \leq \delta^-_{DU} \leq \delta^-_{DU\nu} \\
& 0 \leq \delta^+_{GGR} \leq \delta^+_{GGR\nu} \\
& 0 \leq \delta^-_{GGR} \leq \delta^-_{GGR\nu},
\end{aligned}$$

given the dynamic equation (3.2), and the initial condition e_0. In the expression above $\delta^+_{DU\nu}$, $\delta^-_{DU\nu}$, $\delta^+_{GGR\nu}$ and $\delta^-_{GGR\nu}$ are the veto thresholds for the positive and negative deviations from the two goals: the social planner needs to find the optimal solution satisfying these constraints on the deviations δ^+_{DU}, δ^-_{DU}, δ^+_{GGR} and δ^-_{GGR}, respectively. The aspirations levels can be chosen freely by the social planner, in accordance to some feasibility and/or desirability criterion. In our case the most natural choice is letting the aspiration levels coincide with each goal independently considered, that is:

$$\begin{aligned}
G_{DU} &= \max_{c_t} \int_0^\infty \frac{c_t^{1-\sigma} e_t^{\beta(1-\sigma)} - 1}{1-\sigma} e^{-\rho t} dt \\
& \quad s.t.\ \dot{e}_t = r e_t \left(1 - \frac{e_t}{e^c}\right) - c_t \\
G_{GGR} &= \max_{c_t} \lim_{t \to \infty} \frac{c_t^{1-\sigma} e_t^{\beta(1-\sigma)} - 1}{1-\sigma} \\
& \quad s.t.\ \dot{e}_t = r e_t \left(1 - \frac{e_t}{e^c}\right) - c_t
\end{aligned}$$

In order to apply our GP model, we need to define the satisfaction functions $F_i(\delta^+_i)$, $F_i(\delta^-_i)$. For sake of simplicity we assume that the shape and the properties of the satisfaction functions are the same with respect to both positive and negative deviations, and identical for each criterion. An appropriate satisfaction function may take the following form (Martel and Aouni, 1990):

$$F_\alpha(\delta) = \frac{1}{1 + \alpha^2 \delta^2}$$

Indeed, such a specification implies that: $F(0) = 1$, $F(\infty) = 0$; $F''(\delta) = 0$ if and only if $\delta = \frac{1}{2\alpha}$; $0.9 \leq F(\delta) \leq 1$ if $\delta \leq \frac{1}{3\alpha}$; $0 \leq F(\delta) \leq 0.1$ if $\delta \geq \frac{3}{\alpha}$. This means that this particular function allows to achieve a level of satisfaction between 90% and 100% when $\delta \leq \frac{1}{3\alpha}$, and a level of satisfaction between 0% and 10% when $\delta \geq \frac{3}{\alpha}$. With these properties in mind, we can now proceed to define two threshold levels: the indifference threshold, δ_{ind}, and the dissatisfaction threshold, δ_{dis}. A natural choice for these threshold values is the following: $\delta_{ind} = \frac{1}{3\alpha}$ and $\delta_{dis} = \frac{3}{\alpha}$. Similarly, we need to identify the size of the veto threshold, which we assume to be given by $\delta_v = 2 * \delta_{dis} = \frac{6}{\alpha}$. Note that the parameter α plays a critical role in our satisfaction function, $F_\alpha(\delta)$, and in particular it allows us to obtain different indifference, dissatisfaction and veto thresholds by simply changing its value. It thus may be convenient to compare our results for different choices of this parameter, for example for $\alpha = 0.1$, $\alpha = 0.01$ and $\alpha = 0.001$. It is possible to show that our conclusions are not significantly affected by this parameter value, and thus in what follows we set it equal to 0.1 for the sake of simplicity (see Aouni et al., 2013).

Now we can proceed with the discretization of our problem. First of all, G_{DU} and G_{GGR} in a discrete framework read as follows:

$$\begin{aligned}
G_{DU} = \max_{c_t} \quad & \sum_{t=0}^{T} \frac{c_t^{1-\sigma} e_t^{\beta(1-\sigma)} - 1}{1-\sigma} e^{-\rho t} \\
s.t. \quad & e_{t+1} - e_t = r e_t \left(1 - \frac{e_t}{e^c}\right) - c_t \quad \forall t = 0, \dots . T-1 \\
& 0 = r e_T \left(1 - \frac{e_T}{e^c}\right) - c_T \\
& e_0 \text{ given} \\
G_{GGR} = \max_{c_T} \quad & \frac{c_T^{1-\sigma} e_T^{\beta(1-\sigma)} - 1}{1-\sigma} \\
s.t. \quad & 0 = r e_T \left(1 - \frac{e_T}{e^c}\right) - c_T
\end{aligned}$$

Note that the values G_{DU} and G_{GGR} correspond to the cases $\theta = 1$ and $\theta = 0$ in equation (3.6), respectively. As before, some comments are needed: there is no dynamics after T, and the weights are set as $w_{DU} = \theta$ and $w_{GGR} = 1 - \theta$. The GP problem consists of maximizing the social planner's satisfaction, Z, and in its entire form it reads as follows:

$$\max \quad Z = \frac{\theta}{1 + (\alpha \delta_{DU}^{+})^2} + \frac{\theta}{1 + (\alpha \delta_{DU}^{-})^2} + \frac{1-\theta}{1 + (\alpha \delta_{GGR}^{+})^2} + \frac{1-\theta}{1 + (\alpha \delta_{GGR}^{-})^2} \tag{3.10}$$

$$\sum_{t=0}^{T} \frac{c_t^{1-\sigma} e_t^{\beta(1-\sigma)} - 1}{1-\sigma} e^{-\rho t} - \delta_{DU}^{+} + \delta_{DU}^{-} = G_{DU} \tag{3.11}$$

$$\frac{c_T^{1-\sigma} e_T^{\beta(1-\sigma)} - 1}{1-\sigma} - \delta_{GGR}^{+} + \delta_{GGR}^{-} = G_{GGR} \tag{3.12}$$

$$e_{t+1} - e_t = r e_t \left(1 - \frac{e_t}{e^c}\right) - c_t \quad \forall t = 0, \dots . T-1 \tag{3.13}$$

$$0 = r e_T \left(1 - \frac{e_T}{e^c}\right) - c_T \tag{3.14}$$

$$e_0 \text{ given} \tag{3.15}$$

$$0 \le \delta_{DU}^{+} \le \delta_{DU\nu}^{+} \tag{3.16}$$

$$0 \le \delta_{DU}^{-} \le \delta_{DU\nu}^{-} \tag{3.17}$$

$$0 \le \delta_{GGR}^{+} \le \delta_{GGR\nu}^{+} \tag{3.18}$$

$$0 \le \delta_{GGR}^{-} \le \delta_{GGR\nu}^{-}. \tag{3.19}$$

The results of our simulations are presented in Figure 3.3 and in Figure 3.4. The optimal time paths of consumption and natural resources are qualitatively similar to those found earlier with the scalarization approach. The dashed line still represents the green golden rule ($\theta = 0$) case, in which no dynamics from 0 to T is present. On the left panel of Figure 3.3 we can see that consumption is generally decreasing over time (apart in the $\theta = 1$ case), and it increases

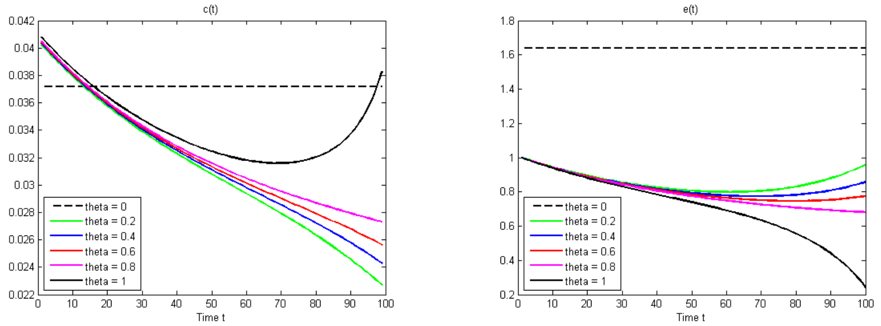

Figure 3.3: Goal programming: evolution of consumption (on the left) and natural resources (on the right) for different values of θ.

with θ. The other side of the same coin can been seen on the right panel of Figure 3.3: the more emphasis we put on discounted utilitarianism, the less resources are left for the long run outcome, with the extreme case being represented by the discounted utilitarian approach in which no resources are left at the end of the time horizon. Since the results qualitatively coincide, their interpretation is exactly equivalent to what discussed earlier concerning the scalarization technique.

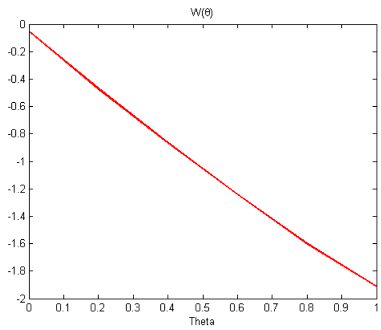

Figure 3.4: Goal programming: social welfare as a function of θ.

Figure 3.4 shows the different levels of social welfare (as in equation (3.6)), W, for values of θ ranging from 0 to 1. Since the time paths of consumption and natural resources are similar to those obtained with the scalarization approach, then it is also reasonable to expect that the behavior of the social welfare function will be equivalent to what discussed earlier. This is confirmed in Figure 3.4, from which we can see that social welfare is decreasing in θ, and thus it achieves its maximum when $\theta = 0$, corresponding to the green golden rule criterion. Even with a GP approach, our results confirm that the green golden rule is not only the preferred criterion from a sustainability point of view but also from a welfare maximization standpoint.

3.5 Conclusion

In (macro)economics literature the notion employed in order to define social welfare is critical, especially whenever we wish to take into account also sustainability issues. This is due to the fact that the most commonly used criterion, namely the discounted utlitarianism, attaches less weight to future generations' wellbeing. Alternative criteria, like the green golden rule and the Chinchilnisky criterion, have been proposed in order to overcome this problem. In this paper we assess and compare the outcomes associated to different welfare criteria by simply relying on a multicriteria model. We consider two alternative specification of the multicriteria problem, based on the scalarization and GP technique respectively, in order to fully analyze the problem in a very simple macroeconomic model with environmental interactions, as in Chichilnisky et al. (1995). For a specific and realistic parametrization of the model, we show that social welfare is a decreasing function of the weight attached to the discounted sum of utilities (θ). Thus, it is possible to identify a clear ranking of criteria in terms of social welfare achievements: the green golden rule leads to the highest level of welfare, the Chichilnisky criterion to an intermediate level, and the discounted utilitarianism to the lowest welfare level. Both the scalarization and the GP techniques, even if relying on different multicriteria philosophies, lead to the same results (see Figure 3.2 and Figure 3.4). Such concordant conclusions corroborate a result that is absolutely not obvious a priori: the green golden rule does have to be preferred to the discounted utilitarianism not only because of its ability to take into account sustainability issues, but also because it allows to achieve higher social welfare levels.

To the best of our knowledge, ours is the first paper trying to assess the implications of different notions of social welfare on the economic and environmental trade off. It is thus natural to wonder to what extent the simplicity of our model specification affects our results. Specifically, in our model natural resources are only used for consumption purposes; however, this is quite a strong simplification of reality. Natural resources represent also an important production factor, without which it may not be possible at all to produce any consumable good. Moreover, production activities, apart from using natural resources, may even be detrimental for the environment through the pollution channel. Furthermore, pollution may generate perverse effects on the production possibilities, thus the nexus between economic activities and environmental dynamics is definitely more complex that what we have considered thus far. Extending the analysis along these directions is needed in order to obtain a wider comprehension of sustainability issues and their implications on social welfare. This is left for future research.

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Chapter 4

Pollution Control under Uncertainty and Sustainability Concerns

4.1 Introduction

Economic activities give rise to several environmental problems and how to regulate such an economic and environmental trade-off is still nowadays a critical open question. One of the issues which has attracted the largest interest in literature is linked to how to optimally control pollution. After decades of researches and debates, it is now clear to both academics and policymakers that regulating polluting activities is all but trivial. This is due to the fact that pollution contributes to several environmental problems, like those related to its transnational diffusion (Highton and Webb, 1981; Ansuategi, 2003) and climate change (Nordhaus, 1982; Bollen and Brink, 2014), but it might also generate economic benefits, like increasing competition and promoting technological progress (Porter and van der Linde, 1995; Otto and Reilly, 2008). Despite the very large body of studies that can be found in literature, two aspects of the pollution control problem have been only marginally analyzed thus far: the implications of uncertainty on pollution and environmental policy[1], and its relation with sustainability and intertemporal equity. This paper tries exactly to fill these gaps by developing a pollution control model which might help policymakers to make better decisions in the determination of the optimal environmental policy in a stochastic framework with rising sustainability concern.

The pollution control problem is quite dated and it consists of determining the optimal policy intervention in order to minimize the social costs (Bawa, 1975) or alternatively maximize the social benefits (Forster, 1975) associated with economic activities, by taking into account both economic and environmental effects. Some earlier studies include Forster (1972), and Keeler et al. (1973), while more recent works are represented by van der Ploeg and Withagen (1991), Athanassoglou and Xepapadeas (2012), and Saltari and Travaglini (2014). With the exception of Athanassoglou and Xepapadeas (2012) who develop a quite sophisticated and cumbersome robust pollution control model under Knightian uncertainty, all the aforementioned papers consider pollution to be perfectly known and

[1]Note that the topic has been frequently analyzed from an empirical point of view, but very rarely considered from a theoretical standpoint. Among empirical works on uncertainty and environment, see recently Yoon and Ratti (2011) and Antonakakis et al. (2014). On the theoretical side, even if with goals substantially different from ours, see Athanassoglou and Xepapadeas (2012).

deterministic. This is obviously a very strong simplification of reality in which, due to the uncertainty surrounding environmental and ecological dynamics, very little is known about the evolution of pollution. Since several developing countries are nowadays experiencing substantial increases in their income levels, accompanied by dramatic increases in emissions, energy demand and use of natural resources (Olivier et al., 2012; U.S. EIA, 2014), the question about how to determine environmental policy in a stochastic context is more relevant than ever. As a preliminary attempt to analyze this issue[2], we develop a simple model of finite horizon pollution control subject to random shocks. Differently from Athanassoglou and Xepapadeas (2012), we do not focus either on the Knightian concept of uncertainty or on society's response to the worst-case scenario. Thus, the work most similar to ours is Saltari and Travaglini's (2014), which analyzes a (deterministic) pollution control problem over a finite horizon. Differently from them, we do not focus on emission constraints and our objective function represents the social costs of pollution. This specific setting allows us to develop a tractable framework to analyze the impact of uncertainty on environmental policy and pollution dynamics.

Apart from its implications on economic outcomes, the (optimal) regulation of polluting activities has also important implications on our ability to eventually achieve a sustainable development pathway. In fact sustainable development clearly requires us to ensure a certain equity across present and future generations (WCED, 1987), not only in terms of economic opportunities but also in terms of environmental quality. In the sustainability literature, the traditional economic discounted utilitarian approach has often been criticized for its inability to take into account the welfare consequences of our today's actions on (very) future generations[3] (Chinchilnisky et al., 1995). Some alternatives have been proposed in order to formally allow also future generations to be considered in the planning problem (Marsiglio, 2011). Chinchilnisky (1997) proposes to modify the objective function in order to accompany the discounted sum of utilities with a long run utility level. In order to formally include in our analysis a certain degree of concern for sustainability issues and future generations we follow Chinchilnisky's (1997) approach[4] and accompany discounted instantaneous costs with an end-of-planning-horizon cost. We wish to understand how environmental policy and pollution are related to the increases in the degree of sustainability concern (representing the weight attached to the end-of-planning-horizon cost) that we are currently witnessing in industrialized economies.

This brief paper proceeds as follows. Section 4.2 presents our model, which consists of a finite horizon pollution control problem in which the stock of pollution is subject to random shocks, and the planner cares for future generations and the level of pollution they will have to bear. In Section 4.3 we explicitly solve the stochastic optimization problem and we characterize the optimal policy and the optimal dynamics of pollution, showing how they are affected by different degrees of sustainability concern and different degrees of uncertainty (in a specific limiting case of our model). In Section 4.4 we present a calibration of our model based on CO_2 data at world level to support our analysis. We thus

[2] Very few papers analyze economic dynamic in a framework of stochastic pollution (Kijima et al., 2011; Saltari and Travaglini, 2011; and Privileggi and Marsiglio, 2013). However, differently from our goals, these works either do either not focus on environmental policy (Privileggi and Marsiglio; 2013), or take it exogenously given (Kijima et al., 2011) or assume the evolution of pollution to be completely exogenous (Saltari and Travaglini, 2011).

[3] Specifically, the presence of a positive discount factor (a necessarily requirement of any infinite horizon optimal control problem) is the source of the problem. Indeed, a positive discount factor means that less and less weight is attached to generations further away in the future, thus the notion of intertemporal equity is automatically ruled out.

[4] Strictly related to our approach, even if with different objectives and methodologies, see recently Colapinto et al. (2015). They propose a multicriteria model in order to assess the implications of different degrees of sustainability concern on the optimal dynamics of economic policy and natural resources.

illustrate the predictions of our model under a realistic parametrization and we are able to assess how different degrees of uncertainty may affect environmental policy and pollution even in our more general setup. Section 4.5 contains concluding remarks and highlights directions for future research. All mathematical technicalities are included in the Appendices 4.6, 4.7 and 4.8.

4.2 The Model

We consider a model of pollution control over a finite horizon in which the stock of pollution is subject to random shocks. The economic framework is very simple: economic agents at each instant of time consume completely[5] their disposable income: $c_t = (1 - \tau_t) y_t$, where c_t denotes consumption, y_t income and $\tau_t \in (0, 1)$ the tax rate. Since economic activity generates pollution as a side product, the tax revenue is used to limit pollution accumulation. Thus, an increase in τ reduces pollution but at the same time lowers consumption possibilities, identifying thus a clear trade-off between economic and environmental performance.

The social planner wishes to minimize the social cost of pollution p_t, by choosing the optimal level of the policy instrument, τ_t. The social cost function, $\mathcal{C}$, is the weighted sum of two different terms: the expected discounted ($\rho > 0$ is the rate of time preference) sum of instantaneous losses generated by economic activities, and the discounted environmental damage associated with the remaining level of pollution at the end of the planning horizon, T. The instantaneous loss function, $c(p_t, \tau_t)$, taking into account both environmental (p_t) and economic (τ_t) costs, is assumed to be increasing and convex in both of its arguments, penalizing deviations from the no-pollution scenario (i.e., $p_t = 0$) and the strength of the policy instrument; for analytical tractability, such a function is assumed to take the following form: $c(p_t, \tau_t) = \frac{p_t^2 (1+\tau_t)^2}{2}$. The damage function, $d(p_T)$, is assumed to be increasing and convex as follows: $d(p_T) = \frac{p_T^2}{2}$. Pollution is a stock variable which increases with flow emissions generated by economic activity and decreases according to the rate of natural pollution absorption; economic output, y_t, generates emissions which increase the stock of pollution at a rate $\eta > 0$, while the natural rate of pollution decay is denoted by $\delta > 0$. The amount of pollution associated with economic activity can be reduced by economic regulation, and one unit of output invested in environmental preservation reduces one unit of pollution; it then follows that the dynamics of pollution under economic regulation is given by the following linear differential equation: $\dot{p}_t = [\eta(1 - \tau_t) - \delta]\, p_t$. The policy instrument τ_t thus represents an environmental tax used to decrease the environmental inefficiencies of economic activities (i.e., the human-induced growth rate of pollution η). The previous differential equation describes the evolution of pollution in absence of uncertainty (as traditionally assumed in the pollution control literature); however, we allow for pollution to be subject to random shocks, assumed to be driven by a geometric Brownian motion.

The social planner needs to choose τ_t in order to minimize the expected social cost function, given the evolution

[5]Note that we abstract from capital accumulation for the sake of simplicity. Allowing for capital accumulation as in van der Ploeg and Withagen (1991) will substantially complicate the analysis. Even in its current form the problem is all but trivial (see Proposition 4), thus extending the analysis to consider the dynamic evolution of capital will make the search for an explicit solution of the Hamilton-Jacobi-Bellman equation even harder. It seems convenient to start the analysis of uncertainty related issues in the simplest possible pollution control problem.

of pollution and its initial (deterministic) condition. The planner's problem can be summarized as follows:

$$\min_{\tau_t} \quad \mathcal{C} = \mathbb{E}\left[\theta \int_0^T \frac{p_t^2(1+\tau_t)^2}{2} e^{-\rho t} dt + (1-\theta)\frac{p_T^2}{2} e^{-\rho T}\right] \tag{4.1}$$

$$s.t. \quad dp_t = [\eta(1-\tau_t) - \delta]\, p_t dt + \sigma p_t dW_t \tag{4.2}$$

$$p_0 > 0 \text{ given,} \tag{4.3}$$

where $\sigma \geq 0$ is the standard deviation of pollution and dW_t the increment of a Wiener process. The parameter $\theta \in [0,1]$ in equation (4.1) measures the relative importance assigned by the social planner to the sum of instantaneous losses rather than the final environmental damage. Note that such a specification is consistent with the notion of sustainability, requiring to ensure a certain degree of intergenerational equity. Specifically, equation (4.1) reflects the so-called Chinchilnisky's criterion which proposes to consider a weighted average between the discounted sum of instantaneous costs and the long run cost associated with pollution (Chinchilnisky, 1997). For smaller values of θ, more emphasis is placed on future generations thus the social planner is more inclined to reduce current pollution (at the expense of reductions in current consumption) in order to leave the posterity with a cleaner environment. Since $c_t = (1-\tau_t)y_t$, any attempt to lower emissions in order to reduce the stock of pollution (rising τ_t) requires to sacrifice some consumption, clearly reflecting the economic and environmental trade-off associated with economic development.

4.3 The Optimal Policy

For the sake of analytical tractability, we consider an equivalent but slightly different formulation of the above stochastic problem, namely:

$$\min_{\tau_t} \quad \mathcal{C} = \mathbb{E}\left[\int_0^T \frac{p_t^2(1+\tau_t)^2}{2} e^{-\rho t} dt + \frac{(1-\theta)}{\theta}\frac{p_T^2}{2} e^{-\rho T}\right] \tag{4.4}$$

$$s.t. \quad dp_t = [\eta(1-\tau_t) - \delta]\, p_t dt + \sigma p_t dW_t \tag{4.5}$$

$$p_0 > 0 \text{ given,} \tag{4.6}$$

Solving this stochastic problem requires to find an explicit expression for the value function solving the Hamilton-Jacobi-Belllman (HJB) equation associated with the problem (4.4), (4.5) and (4.6). After some algebra it is possible to claim the following.

Proposition 4. *The value function associated with the problem (4.4), (4.5) and (4.6) is given by:*

$$\mathcal{J}(t, p_t) \quad = \quad \frac{1}{2} p_t^2 V_t e^{-\rho t}, \tag{4.7}$$

where V_t is the solution of the following differential equation:

$$\dot{V}_t \quad = \quad V_t^2 \eta^2 + V_t[\rho - 2(\eta - \delta) - \sigma^2] - 1, \tag{4.8}$$

with the boundary condition $V_T = \frac{1-\theta}{\theta} \geq 0$. *Assume that:*

$$\theta \in (\underline{\theta}; \overline{\theta}) \tag{4.9}$$

where $\underline{\theta} \equiv \frac{2\eta^2}{2\eta^2+2(\eta-\delta)-\rho+\sigma^2+\sqrt{[2(\eta-\delta)+\sigma^2-\rho]^2+4\eta^2}}$ *and* $\overline{\theta} \equiv \frac{2\eta^2}{2\eta^2+2(\eta-\delta)-\rho+\sigma^2-\sqrt{[2(\eta-\delta)+\sigma^2-\rho]^2+4\eta^2}}$*; then the optimal rule for the taxation rate,* τ_t^* *and the optimal dynamic path of pollution are respectively given by:*

$$\begin{aligned}
\tau_t^* &= \frac{1}{2\eta}\Bigg\{2(\eta-\delta)-\rho+\sigma^2+\sqrt{[2(\eta-\delta)+\sigma^2-\rho]^2+4\eta^2}\tanh\Bigg[\frac{\sqrt{[2(\eta-\delta)+\sigma^2-\rho]^2+4\eta^2}(T-t)}{2}+ \\
&\quad +\operatorname{arctanh}\left(\frac{2(1-\theta)\eta^2-2(\eta-\delta)\theta+\rho\theta-\sigma^2\theta}{\theta\sqrt{[2(\eta-\delta)+\sigma^2-\rho]^2+4\eta^2}}\right)\Bigg]\Bigg\} && (4.10) \\
p_t^* &= p_0\exp\left\{\int_0^t\left[\eta(1-\tau_s^*)-\delta-\frac{1}{2}\sigma^2\right]ds+\sigma W_t\right\} && (4.11)
\end{aligned}$$

where $\tanh(z) = \frac{e^z-e^{-z}}{e^z+e^{-z}}$ *and* $\operatorname{arctanh}(z) = \frac{\log(1+z)-log(1-z)}{2}$, *with* $-1 < z < 1$, *are the hyperbolic tangent function and its inverse, respectively.*

Proof. See Appendix 4.6. ∎

Note that in order for the optimal level of taxation τ_t^* to be well defined, the inverse hyperbolic tangent functions needs to be well defined too, and this happens whenever the value of the parameter θ falls in the interval $\theta \in (\underline{\theta}; \overline{\theta})$ as specified in equation (4.9). From now onwards we proceed by assuming that such a condition holds. Proposition 4 clearly shows that the optimal level of taxation is not constant, and as a result the trend of the pollution stock (even in a purely deterministic framework) is time-varying too. The same comment applies to the trend of pollution stock, which (even in absence of shocks) tends to change as a result of the time evolution of the optimal taxation level. From equation (4.10) we can note that the optimal policy, determining the amount of resources diverted from economic to environmental activities, strictly depends upon the planner's degree of sustainability concern (i.e., $1-\theta$). The degree of sustainability concerns indirectly (through the optimal taxation channel) affects also the dynamics of pollution (see equation (4.11)), and thus it is natural to wonder whether the increasing sustainability concern that we are currently experiencing (at least within industrialized countries) is going to generate positive or negative consequences on the amount of pollution our society will have to bear in the long run. Despite the quite complex expression for τ^*, it is possible to show that the following result holds.

Proposition 5. *Provided that* $\theta \in (\underline{\theta}; \overline{\theta})$ *holds, the optimal taxation level (i.e.,* τ_t^**) increases with the degree of sustainability concern (i.e.,* $1-\theta$*).*

Proof. See Appendix 4.7. ∎

Proposition 5 shows the existence of a positive relationship between the optimal level of (environmental) policy intervention and the degree of sustainability concern. Intuitively, this result suggests that the more the planner cares for sustainability issues, the more convenient it will be to actively intervene in order to limit the amount of pollution the society will have to bear in the long run. As a result, the level of taxation will be larger dampening economic activities and simultaneously reducing the stock of pollution. This suggests that the current trend of a growing environmental and sustainability concern might be effective in achieving a more sustainable development path in the long run, but such an increased sustainability will occur at the cost of reductions in consumption opportunities. However, promoting further increases in the degree of sustainability concern, through environmental education, sensibilization and green campaigns, might be a valuable tool for supporting a greener and more sustainable future.

The above results are all related to relationship between the degree of sustainability concern and economic policy, and we have not analyzed what role uncertainty plays in this context. However, because of the complex expression for τ_t^*, it is possible to derive only a sufficient condition ensuring that τ_t^* is monotonically related to σ^2, and specifically it monotonically increases with σ^2. This allows us to state the following result.

Proposition 6. *Provided that $\theta \in (\underline{\theta}; \overline{\theta})$ holds, the optimal taxation level (i.e., τ_t^*) increases with the degree of uncertainty (i.e., σ^2) whenever $\sigma^2 \leq \rho - 2(\eta - \delta) - \frac{2\theta}{1-\theta}$.*

Proof. See Appendix 4.8. ■

Proposition 6 shows that $\frac{\partial \tau_t^*}{\partial \sigma^2}$ turns out to be undoubtedly positive if a certain condition holds, while nothing can be explicitly said whenever such a condition is not met. Note that for $\theta \in [0, 1]$ the condition above can hold only for very small values of θ. In fact, whenever $\theta \to 0$, the condition reads as $\sigma^2 \leq \rho - 2(\eta - \delta)$, which (provided that ρ is sufficiently large) identifies a threshold value for the uncertainty parameter below which increases in uncertainty undoubtedly lead to increases in the optimal level of taxation. The case $\theta = 0$ is a very extreme case representing a situation in which the degree of sustainability concern is maximal and thus social costs are defined according to the green golden criterion (Chinchilnisky et al., 1995). In our setup such a criterion states that only the long run costs should be considered in order to determine the level of policy intervention. Intuitively, in such a framework higher levels of pollution stock are definitely undesirable and since the economic costs associated with pollution reduction are not considered (see how the social cost function (4.1) would read whenever $\theta \to 0$), with rising uncertainty it is clearly convenient to firmly intervene in order to limit as much as possible the pollution stock[6]. However, apart from this very special limiting case, the condition in Proposition 6 cannot be realistically met thus nothing can be said from an analytical point of view on the role of uncertainty in our general setting.

4.4 A Calibration Based on Global CO_2 Data

In order to shed some more light on this relationship between uncertainty and policy intervention and to illustrate the implications of different degrees of sustainability concern on environmental policy, we rely on a calibration based on global CO_2 data. In order to obtain an estimate of our parameter values, we need first of all to consider that the dynamic stochastic equation describing the evolution of pollution over time, namely equation (4.2), suggests an exponential growth for pollution. In order to quantify pollution variations we focus on atmospheric CO_2 concentrations, expressed in parts per million (ppm). We rely on two sets of data about CO_2 levels: the long (2,000-year) record from the Law

[6]Note that, because of what intuitively just discussed, the $\theta = 0$ case represents a degenerate case giving rise to a trivial solution in which the optimal level of policy intervention is always maximal, that is $\tau^* = 1, \forall t$.

Dome ice core in Antarctica, provided by the Carbon Dioxide Information Analysis Center of the U.S. department of energy (Etheridge et al., 1998), and the more recent years time series made available by the Earth System Research Laboratory of the National Oceanographic and Atmospheric Administration (Dlugokencky and Tans, 2015). We rely on the former to obtain concentrations data from the 1750 to 1979, and on the latter for data from 1980 to 2015. By joining these two data sets, we can see that nowadays the global level of CO_2 is about 400 ppm, and the CO_2 concentration at world level has followed an exponential growth pattern since the industrial revolution (see Figure 4.1). This provides some clear support for our formulation of pollution dynamics, as expressed in equation (4.2).

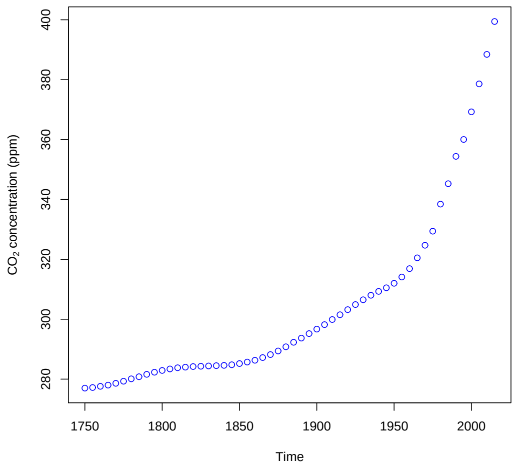

Figure 4.1: Evolution of CO_2 concentrations from the industrial revolution. [Sources: Etheridge et al. (1998) and Dlugokencky and Tans (2015)]

We extrapolate the exponential growth rate from the 1750–2015 time series, obtaining a net (of natural absorption) rate of pollution growth, $\eta - \delta$ equal to 0.001. By following Saltari and Travaglini (2014), we set the natural pollution decay rate, δ equal to 0.05, implying that the rate pollution growth η is equal to 0.051. The value of the standard deviation, σ, has been calculated simply averaging the annual level of standard deviations of the recent data from Dlugokencky and Tans (2015), obtaining 0.164. As traditionally assumed in literature we set the rate of time preference, ρ, equal to 0.04 (Saltari and Travaglini, 2014). The time horizon has been arbitrarily set at 30 years, but it is possible to show that even extending the time frame does not qualitatively modify our results. The initial value of the pollution stock, p_0, is set equal to the current (2015) level of CO_2 concentration, that is 400.23 ppm. We thus consider the following parameter values: $\eta = 0.051$, $\delta = 0.05$, $\sigma = 0.164$, $\rho = 0.04$, $p_0 = 400.23$ and $T = 30$. We allow for different values of θ in order to show how economic policy and pollution stock vary with different degrees of sustainability concern (represented by $1 - \theta$). Specifically, we consider three different values of θ, representing a low ($\theta = 0.9$), medium ($\theta = 0.5$) and high ($\theta = 0.1$) degree of sustainability concern.

The outcome of our calibration is illustrated in Figure 4.2, in which we first consider a purely deterministic framework (i.e., $\sigma = 0$) and we represent the dynamic evolution of the optimal taxation rate (on the left panel) and the pollution stock (on the right panel). It is clear that τ^* monotonically falls with θ while p^* monotonically rises with

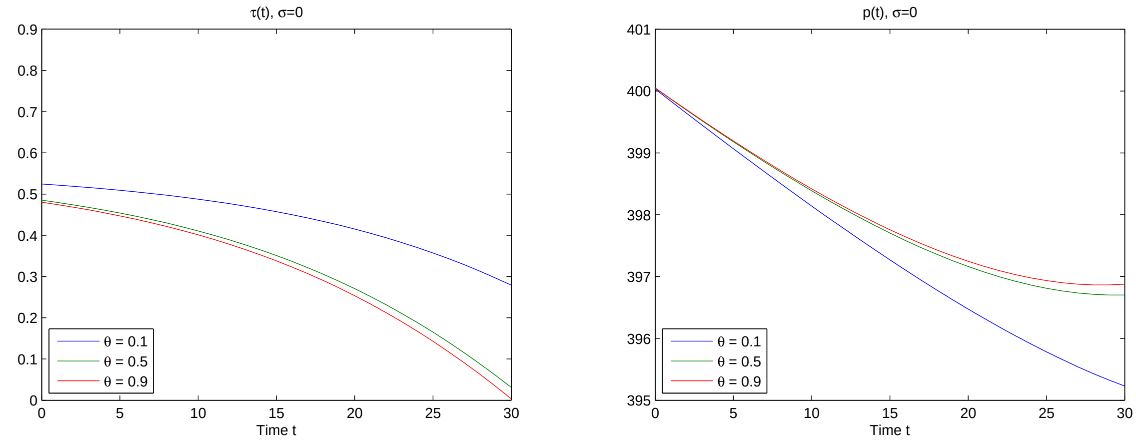

Figure 4.2: Deterministic case: dynamic evolution of the optimal level of taxation τ_t^* (left panel) and pollution p_t^* (right panel) as a function of the degree of sustainability concern, θ.

θ (these monotonicity results are robust even considering a wider values of θ). These diametrically different effects of the degree of sustainability concern on the taxation rate and pollution stock are due to the negative relationship between p^* and τ^* (see equation (4.11)). What these results show is that the larger the weight attached to the long run level of pollution (the lower θ), the stricter the optimal environmental policy (the higher τ_t^*) and thus the healthier the environment (the smaller p_t^*).

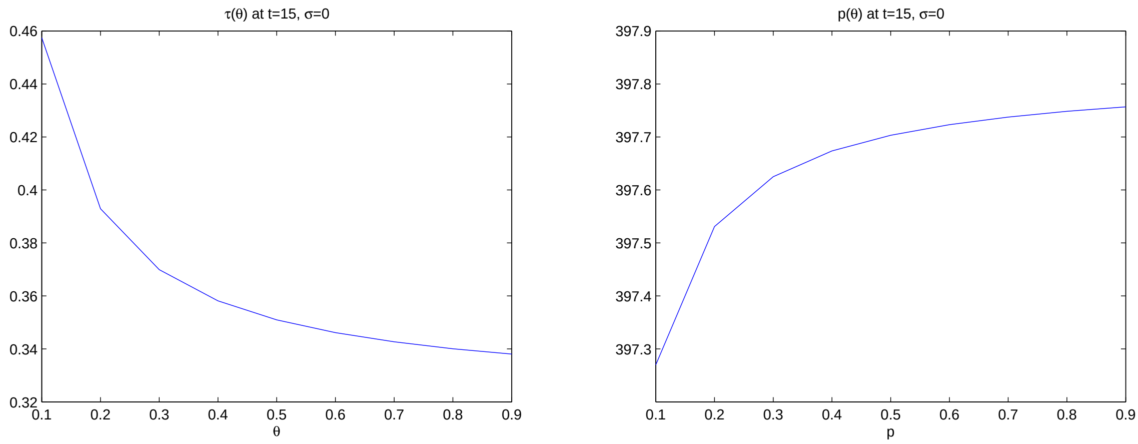

Figure 4.3: Optimal level of taxation τ^* (left) and consequent level of pollution p^* (right) as functions of the degree of sustainability concern θ, at a given point in time (specifically, $t = 15$).

In order to better understand the impact of different degrees of sustainability concern on economic policy and environmental outcomes, it might be convenient to fix one point in time, $t = \widetilde{t}$ (e.g., $\widetilde{t} = \frac{T}{2}$) for a while. This allows us to assess to what extent in a purely static framework a different θ is going to affect the taxation and pollution levels. As clearly shown by Figure 4.3, the optimal level of taxation decreases at its fastest pace with low values of θ, while the change in τ is barely evident for larger values. This suggests that increases in the degree of sustainability concern (a decrease in θ) may have relevant effects only whenever the society (i.e., the social planner) does not care enough about sustainable outcomes, since whenever the care for the long run outcome is already high further decreases in θ may have

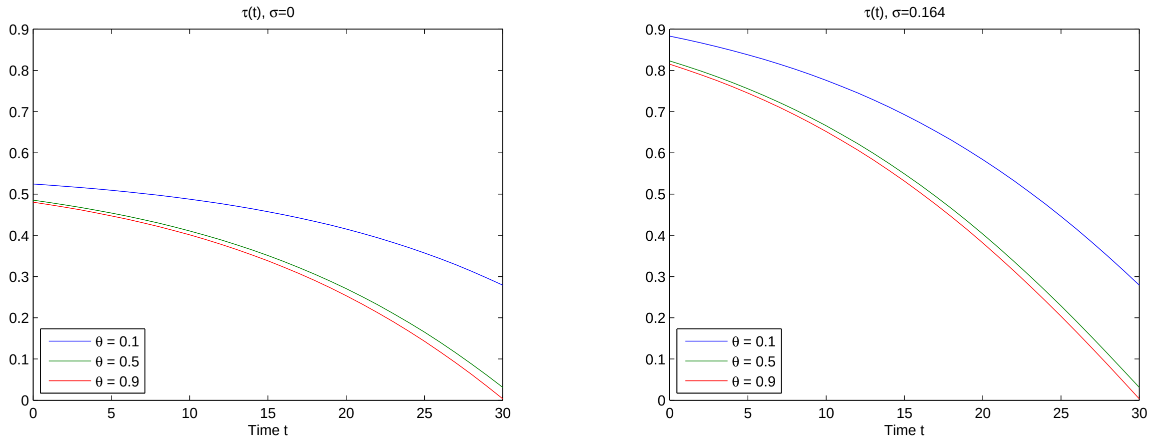

Figure 4.4: Comparison between the deterministic (left, $\sigma = 0$) and the stochastic (right, $\sigma = 0.164$) scenarios.

only negligible effects. This suggest the existence of a threshold value determining the effectiveness of policies aiming to eventually promote increases in the degree of sustainability concern. Indeed, the degree of sustainability concern has to be above a certain threshold to actually translate into a leap of policy intervention. Accordingly, by looking at the right panel of Figure 4.3, we can see that the pollution stock decreases substantially only when the degree of sustainability concern is above a certain threshold (that is θ is substantially small), boosting policy intervention and consequently curbing the accumulation of pollution.

The conclusions that we have discussed thus fare are all derived from a deterministic framework, thus we might be wondering whether such results still hold also when uncertainty is taken into account. In Figure 4.4 we compare the evolution of the taxation rate and pollution stock in a stochastic (left panels) and deterministic (right panels) contexts. Despite the fact that for all the θ values considered the sufficient condition in Proposition 6 does not hold, the optimal taxation in the stochastic case is always greater than the deterministic one, consistently with a precautionary motive (Athanassoglou and Xepapadeas, 2012). This states that under a realistic model's parametrization, environmental costs outweigh economic costs such that with a higher uncertainty in pollution dynamics it is convenient to adopt stricter policy measures in order to minimize the social costs.

Even if it is true that τ in the stochastic case is always greater than the deterministic one, by considering the same level of θ, it is also possible to note that the difference in the optimal taxation between the two scenarios decreases as time goes by, meaning that the effect of uncertainty on the optimal policy path decreases over time. This can be explained by the fact that an optimal policy intervention reduces the impacts of uncertainty on the pollution stock, such that in the very long run its level is determined for the largest extent by the degree of sustainability concern. This can be seen in the left panel of Figure 4.5 which shows the time evolution of the difference in the optimal taxation between the stochastic and the deterministic case for a certain value of the degree of sustainability concern, that is $\theta = 0.5$. Moreover, as it is possible to note from the right panel of Figure 4.5, which focuses on the same difference at $t = 0$, at the beginning of the time interval the difference in taxation between the stochastic and deterministic case are larger the smaller the value of θ, implying that the uncertainty induced (economic) cost associated to the optimal policy is higher the smaller θ, that is the higher the degree of sustainability concern.

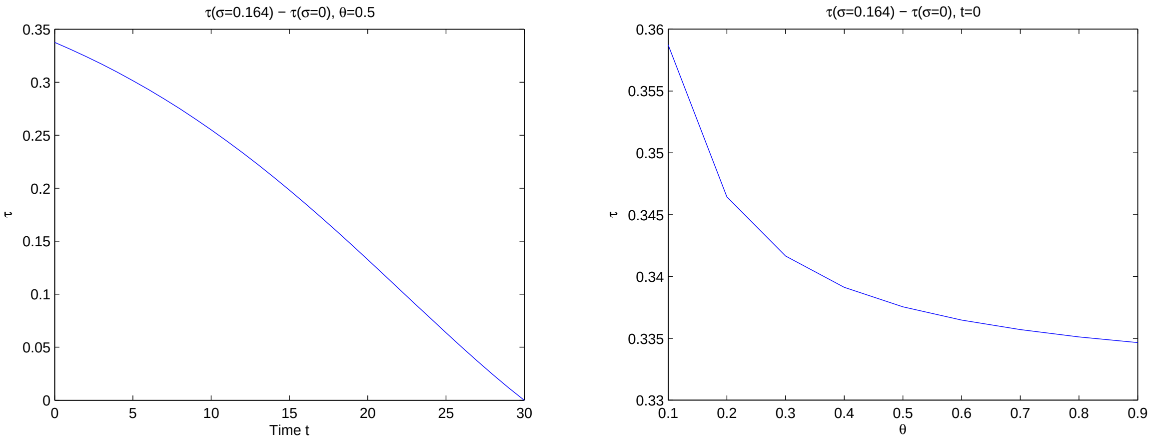

Figure 4.5: Time evolution (left) and initial period (right) differences in the optimal taxation levels between the stochastic and deterministic scenarios.

4.5 Conclusions

The rising interests that we are witnessing among policymakers and academics towards sustainable development and the high uncertainty associated with future environmental outcomes naturally raise the question about how environmental policy should be determined in order to take into account such factors. In order to give a preliminary answer to this question we analyze a finite horizon problem of pollution control under uncertainty in which the planner is (partially) moved by sustainability concern. Despite the model's simplicity, the problems turns out to be all but trivial. We show that the optimal level of environmental policy is non-constant and it is clearly affected by both the degree of uncertainty and sustainability concern. Specifically and intuitively, both larger degrees of sustainability concern and larger degrees of uncertainty lead to a stricter environmental policy, reducing thus the environmental burden imposed on the society both in the short and long run. Clearly the degree of sustainability concern may be effectively affected through specific (education or advertising) policies, thus it represents an important tool to achieve a more sustainable and greener future. However, the reduction in the environmental burden associated with pollution control comes at the cost of a reduction in consumption possibilities, thus assessing the net impact on social costs of further increases in the suststainability concern is not straightforward.

This work represents a first attempt to analyze the impact of uncertainty and sustainability issues in a pollution control model, thus the analysis cannot be considered exhaustive. Indeed, for the sake of simplicity we have to introduce some simplifying assumptions which might have limited our model's ability to describe in full the nature of the problem. In particular, since we abstract from capital accumulation we cannot comment on the effects of uncertainty and the degree of sustainability concern on both economic performance and pollution in growing economies. Moreover, the model's setup does not allow to distinguish between the notion of uncertainty and that of risk, thus it is not possible to disentangle their relationships with the degree of sustainability concern. Extending the analysis to allow for economic growth (as in van der Ploeg and Withagen, 1991) and for a Knightian concept of uncertainty (as in Athanassoglou and Xepapadeas, 2012) is left for future research.

4.6 Appendix A: Optimal Solution and Sufficiency

By denoting with $\mathcal{J}(t, p_t)$ the value function associated to our stochastic problem (4.4), (4.5) and (4.6) and by omitting the time subscript for sake of clarity, the HJB equation reads as:

$$-\frac{\partial \mathcal{J}}{\partial t} = \min_{\tau} \left\{ \frac{1}{2} p^2 (1+\tau^2) e^{-\rho t} + [(1-\tau)\alpha - \delta] p \frac{\partial \mathcal{J}}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 \mathcal{J}}{\partial p^2} \right\} \tag{4.12}$$

while the corresponding terminal condition as:

$$\mathcal{J}(T, p_T) = (1-\theta) \frac{1}{2} p_T^2 e^{-\rho T} \tag{4.13}$$

The first order necessary (and sufficient; see below) condition for τ yields:

$$\tau = \frac{\eta e^{-\rho t}}{p} \frac{\partial \mathcal{V}}{\partial p} \tag{4.14}$$

We proceed by guessing the form of the value function and verifying that our guess is correct. Our sophisticated guess is:

$$\mathcal{J}(t, p) = \frac{1}{2} p^2 V e^{-\rho t} \tag{4.15}$$

where V is a variable to be determined. By computing its derivatives:

$$\frac{\partial \mathcal{J}}{\partial t} = \frac{1}{2} p^2 \left[\frac{\partial V}{\partial t} - \rho V \right] e^{-\rho t}, \tag{4.16}$$

$$\frac{\partial \mathcal{J}}{\partial p} = p V e^{-\rho t} \tag{4.17}$$

$$\frac{\partial^2 \mathcal{J}}{\partial p^2} = V e^{-\rho t}, \tag{4.18}$$

and substituting (4.17) into (4.14), we obtain:

$$\tau = \eta V \tag{4.19}$$

By plugging (4.16), (4.17) and (4.18) into (4.12) and simplifying the expression, we obtain the following ordinary differential equation in V:

$$\frac{\partial V}{\partial t} = V^2 \eta^2 + V[\rho - 2(\eta - \delta) - \sigma^2] - 1, \tag{4.20}$$

with the boundary condition $V_T = \frac{1-\theta}{\theta} \geq 0$, from evaluating (4.13) and (4.15) at T. The solution of the above differential equation can be used to derive the path of the optimal tax rate (from (4.19)) and finally the expected path of pollution (from (4.5)). Indeed, by solving (4.20) along with its boundary condition for V_t and substituting into

(4.19) we get the optimal dynamics of the tax rate:

$$\tau^* = \frac{1}{2\eta}\left\{2(\eta-\delta)-\rho+\sigma^2+\tanh\left[\frac{\sqrt{M}(T-t)}{2}+\operatorname{arctanh}\left(\frac{2(1-\theta)\eta^2-2(\eta-\delta)\theta+\rho\theta-\sigma^2\theta}{\theta\sqrt{M}}\right)\right]\sqrt{M}\right\}, \tag{4.21}$$

where $M=[2(\eta-\delta)+\sigma^2-\rho]^2+4\eta^2$, $\tanh(z)=(e^z-e^{-z})/(e^z+e^{-z})$ is the hyperbolic tangent function and $\operatorname{arctanh}(z)=\frac{1}{2}[\log(1+z)-log(1-z)]$, with $-1<z<1$, its inverse. By plugging the above expression in (4.5), which describes a geometric Brownian motion with time-dependent coefficients, it is possible to determine the time evolution of pollution, whose closed form expression is given in equation (4.11).

In order to verify the correctness of our guess, we use the stochastic maximum principle proposed by Framstad et al. (2004) to show that the policy rule identified in (4.21) is optimal. By defining $m_t \equiv \partial\mathcal{J}/\partial p$ and $n_t \equiv \partial^2\mathcal{J}/\partial p^2$, it is possible to rewrite (4.12) as:

$$-\frac{\partial\mathcal{J}}{\partial t} = \min_{\tau}\left\{\frac{1}{2}p^2(1+\tau^2)e^{-\rho t}+[(1-\tau)\eta-\delta]pm+\frac{1}{2}\sigma^2p^2n\right\}=\min_{\tau}\ \mathcal{H},$$

where:

$$\mathcal{H} = \frac{1}{2}p^2(1+\tau^2)e^{-\rho t}+[(1-\tau)\eta-\delta]pm+\frac{1}{2}\sigma^2p^2n \tag{4.22}$$

where denotes the the stochastic Hamiltonian. Theorem 2.1 of Framstad et al. (2004) states that, for an admissible set of state and controls, if the minimized Hamiltonian $\hat{\mathcal{H}}$ (that is the Hamiltonian $\mathcal{H}$ evaluated at the value of the optimal control τ^*) is convex in p for all t in $[0,t]$, then the pair (τ^*,p) represents an optimal pair for the problem. Note that $\mathcal{H}$ is strictly convex in τ since $\partial^2\mathcal{H}/\partial\tau^2=p^2e^{-\rho t}>0$. The control which minimizes $\mathcal{H}$ is given by equation (4.14) and so the minimized Hamiltonian is:

$$\hat{\mathcal{H}} = \frac{1}{2}p^2\left(1+\frac{m^2\eta^2}{p^2(e^{-\rho t})^2}\right)e^{-\rho t}+pm\left[\left(1-\frac{m\eta}{pe^{-\rho t}}\right)\eta-\delta\right]+\frac{1}{2}\sigma^2p^2n$$

which is strictly convex in p, since $\partial^2\hat{\mathcal{H}}/\partial p^2=e^{-\rho t}+n\sigma^2>0$.

4.7 Appendix B: Proof of Proposition 5

The derivative of the optimal policy τ^* with respect to θ reads as:

$$\begin{aligned}\frac{\partial\tau_t^*}{\partial\theta} &= \frac{\sqrt{M}}{2}\left\{1-\tanh\left[\frac{\sqrt{M}}{2}(T-t)+\operatorname{arctanh}\left(\frac{B}{\sqrt{M}}\right)\right]^2\right\}\left\{\left(\frac{-2\eta^2-\sigma^2+2\delta-2\eta+\rho}{\theta\sqrt{M}}-\frac{B}{\theta^2\sqrt{M}}\right)*\right.\\ &\quad\left.*\left[\left(1-\frac{B}{\theta^2(\sqrt{M})^2}\right)\eta\right]^{-1}\right\}\end{aligned} \tag{4.23}$$

where $M = [2(\eta-\delta)+\sigma^2-\rho]^2+4\eta^2$ and $B = 2(1-\theta)\eta^2 - 2(\eta-\delta)\theta+\rho\theta-\sigma^2\theta$. The sign of the above derivative is determined by the product of three terms, $\frac{\sqrt{M}}{2}$ and the two terms in the curly brackets. Provided that the hyperbolic tangent function is well defined, as per condition (4.9), the first term, $\frac{\sqrt{M}}{2}$, is clearly non-negative. The second term, namely $\left\{1-\tanh\left[\frac{\sqrt{M}}{2}(T-t)+\operatorname{arctanh}\left(\frac{B}{\sqrt{M}}\right)\right]^2\right\}$, is non-negative too since the hyperbolic tangent takes values in $[-1,1]$. After some algebra, the third term, $\left\{\left(\frac{-2\eta^2-\sigma^2+2\delta-2\eta+\rho}{\theta\sqrt{M}}-\frac{B}{\theta^2\sqrt{M}}\right)\left[(1-\frac{B}{\theta^2(\sqrt{M})^2})\eta\right]^{-1}\right\}$, can be rearranged to obtain:

$$-\frac{\sqrt{\sigma^4-4\delta\sigma^2+4\eta\sigma^2-2\rho\sigma^2+4\delta^2-8\delta\eta+4\delta\rho+8\eta^2-4\eta\rho+\rho^2}}{2[\eta\theta^2(-\eta^2-\sigma^2+2\delta-2\eta+\rho+1)+\eta\theta(2\eta^2+\sigma^2-2\delta+2\eta-\rho)-\eta^3]} \tag{4.24}$$

Since the numerator in (4.24) is clearly non-negative, the sign of its denominator determines the sign of (4.23): if this is positive then the whole derivative will be negative, while it will be positive otherwise. It turns out that the conditions for the hyperbolic tangent function to be well defined, as in equation (4.9), ensure that the denominator of the above expression is positive, such that the sign of (4.23) is overall negative.

4.8 Appendix C: Proof of Proposition 6

From equation (4.10) it is clear that what complicates the determination of the sign of $\frac{\partial\tau^*}{\partial\sigma^2}$ is the argument of the inverse hyperbolic tangent. Indeed, apart from this term whose derivative has an uncertain sign, all other terms suggest the existence of a monotonically increasing relationship between the optimal taxation and the degree of uncertainty. Thus, we can undoubtedly assess the sign of $\frac{\partial\tau^*}{\partial\sigma^2}$ only whenever also the argument of the inverse hyperbolic tangent rises with σ^2. In the following we denote the argument of the inverse hyperbolic tangent with Ω, which from equation (4.10) reads as:

$$\Omega = \frac{2(1-\theta)\eta^2-2(\eta-\delta)\theta+\rho\theta-\sigma^2\theta}{\theta\sqrt{[2(\eta-\delta)+\sigma^2-\rho]^2+4\eta^2}}$$

After some algebra the derivative of the above term with respect to σ^2 yields:

$$\frac{\partial\Omega}{\partial\sigma^2} = \frac{-2\eta^2\left(2\delta\theta-2\eta\theta+\rho\theta-\theta\sigma^2-2\delta+2\eta-\rho+2\theta+\sigma^2\right)}{\theta\left(4\delta^2-8\delta\eta+4\delta\rho-4\delta\sigma^2+8\eta^2-4\eta\rho+4\eta\sigma^2+\rho^2-2\rho\sigma^2+\sigma^2\right)^{\frac{3}{2}}}$$

Since the denominator in above expression is clearly non-negative, the sign of its numerator determines the sign of $\frac{\partial\Omega}{\partial\sigma^2}$: whenever this is negative then the whole derivative will be positive. This happens whenever the condition stated in Proposition 6, $\sigma^2 \leq \rho-2(\eta-\delta)-\frac{2\theta}{1-\theta}$, holds.

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